மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

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தொலைநிலை தொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

B.Sc. MATHEMATICS

I YEAR

ALGEBRA & TRIGONOMETRY

Sub. Code: JMMA11 **(For Private Circulation only)**

MATHEMATICS –I YEAR

JMMA11: ALGEBRA & TRIGONOMETRY

SYLLABUS

Unit I

Reciprocal Equations-Standard form–Increasing or decreasing the roots of a given equation-Approximate solutions of roots of polynomials by Horner's method – related problems.

Unit II

Summation of Series: Binomial– Exponential –Logarithmic series (Theorems without proof) – Approximations - related problems.

Unit III

Characteristic equation – Eigen values and Eigen Vectors - Similar matrices - Cayley - Hamilton Theorem (Statement only) - Finding powers of square matrix, Inverse of a square matrix up to order 3, - related problems.

Unit IV

Expansions of sin nθ, cos nθ in powers of sin θ, cos θ - Expansion of tan nθ in terms of tan θ, Expansions of $\cos^n\theta$, $\sin^n\theta$, $\cos^m\theta \sin^n\theta$ –Expansions of $\tan(\theta_1+\theta_2+,\ldots,+\theta_n)$- related problems.

Unit V

Hyperbolic functions – Relation between circular and hyperbolic functions Inverse hyperbolic functions, Logarithm of complex quantities, - related problems.

Text Book

1. T.K. Manickavasagam Pillai, T. Natarajan and K S. Ganapathy, Algebra Vol-I, S. Viswanathan Publishers and Printers Pvt Ltd 2015.
2. S. Arumugam and A. Thangapandi Issac, Theory of Equations and Trigonometry, New Gamma Publishing House, Palayamkottai. 2006

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Unit I

Reciprocal Equations-Standard form–Increasing or decreasing the roots of a given equation-Approximate solutions of roots of polynomials by Horner's method – related problems.

Reciprocal Equation:

Definition:

If an equation remains unaltered, when x is changed into its reciprocal, it is called a reciprocal equation.

Let $x^n + P_1, x^{n-1} + P_2 x^{n-2} + \cdots + P_{n-1}x + P_n = 0$ ....(1)

be a reciprocal equation. When x is changed into its reciprocal $1/x$, we get the transformed equation $P_n x^n + P_{n-1}x^{n-1} + P_{n-2}x^{n-2} + \cdots . + P_1 x + 1 = 0$

(i.e.) $x^n + \frac{P_{n-1}}{P_n} x^{n-1} + \frac{P_{n-2}}{P_n} x^{n-2} + \cdots + \frac{P_1 x}{P_n} + \frac{1}{P_n} = 0$ ....(2)

Since (1) is a reciprocal equation, it must be same as equation (2)

$$\frac{P_{n-1}}{P_n} = P_1; \frac{P_{n-2}}{P_n} = P_2 \cdots ; \frac{P_1}{P_n} = P_{n-1}, \frac{1}{P_n} = P_n$$
$$P_n^2 = 1 \Rightarrow P_n = \pm 1.$$

Case (i) $P_n = 1$

Then, $P_{n-1} = P_1, P_{n-2} = P_2, P_{n-3} = P_3$

In this case, the coefficients of the terms equidistant from the beginning and the end are equal in magnitude and have the same sign

Case (ii) $P_n = -1$

Then $P_{n-1} = -P_1, P_{n-2} = -P_2$ ....

In this case, the terms equidistant from the beginning and the end are equal in magnitude but different in sign.

Standard form of reciprocal equation:

If α be a root of a reciprocal equation, $1/\alpha$ also be a root, for it is a root of the transformed equation and the transformed equation is identical with the first equation. Hence, the roots of a reciprocal equation occur in pairs $\alpha, 1/\alpha, \beta, 1/\beta$

When the degree is odd, one of its roots must be its own reciprocal

(i.e.) $r = 1/r$

$\Rightarrow r^2 = 1 \Rightarrow r = \pm i$

Definition:

Even degree reciprocal equation with like sign is a S.R.E Result :-

(1) If $f(x) = 0$ is a R.E and odd degree with like sign then $x + 1$ is a factor of $f(x)$

Then, $\frac{f(x)}{x+1}$ is a standard $R \cdot E$

(2) If $f(x) = 0$ is a $R \cdot E$ and odd degree with Unlike sign then x^{-1} is a factor of $f(x)$.

Then, $\frac{f(x)}{x-1}$ is a S.R.E

(3) If $f(x) = 0$ is a S.R.E with even degree with like sign then $f(x)$ is a $S \cdot R \cdot E$

(4) If $f(x) = 0$ is a $S \cdot R \cdot E$ with even degree with unlike sign, dividing by $x^2 - 1$, this reduces to a $R \cdot E$ of like sigh of even degree.

Then $\frac{f(x)}{x^2-1}$ is a $S \cdot R \cdot E$.

Example 1:

Find the roots of the equation $x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0$.

Solution:

Let $f(x) = x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1$

Here, $f(x)$ is the reciprocal eqn with odd degree and like sign.

$\Rightarrow (x + 1)$ is a factor of $f(x)$,

$x = -1$ is a root of $f(x)$

	1	4	3	3	4	1
-1	0	-1	-3	0	-3	-1
	1	3	0	3	1	0

$$\text{f(x)} = (x+1)(x^4+3x^3+3x+1)$$

To find the remaining roots of $f(x)$. It is enough to solve the eq n $(x^4+3x^3+3x+1)=0$ …………..(1)

Divide equation (1) by x^2, we get

$$\Rightarrow x^2+3x+3/x+1/x^2=0$$
$$x^2+1/x^2+3(x+1/x)=0 \quad \ldots\ldots\ldots\ldots..(2)$$

Let $y = x+1/x$.

$$(x+1/x)^2 = y^2$$
$$x^2+1/x^2+2 = y^2$$

$x^2+1/x^2 = y^2-2.$

then equation (2) becomes,

$$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$y^2-2+3y=0.$

$$y = \frac{-3\pm\sqrt{9+8}}{2} = \frac{-3\pm\sqrt{17}}{2}$$

$$x+1/x = \frac{-3+\sqrt{17}}{2}, x+1/x = \frac{-3-\sqrt{17}}{2}$$
$$\frac{x^2+1}{x} = \frac{-3+\sqrt{17}}{2}, \frac{x^2+1}{x} = \frac{-3-\sqrt{17}}{2}$$

$2x^2 + 3x - \sqrt{17}x + 2 = 0, 2x^2 + 3x + \sqrt{17}x + 2 = 0$

These two equation yields the required outs of the given eqn.

Example 2:

Solve the equation $6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$.

Solution:

Let $f(x) = 6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$

Here, $f(x)$ is a reciprocal eq n of odd degree with unlike sign

$\Rightarrow (x - 1)$ is a factor of $f(x)$

$x = H$ is a root of $f(x)$

	6	-1	-4^3	4^3	1	-6
1	0	6	5	-38	5	6
	6	5	-38	5	6	0

$f(x) = (x - 1)(6x^4 + 5x^3 - 38x^2 + 5x + 6)$

To find the remaining root of $f(x)$.

Such that is enough to solve the equation

$6x^2 + 5x^2 - 38x^2 + 5x + 6 = 0$ ……….(2)

Divide equation (1) by x^2

$$6x^2 + 5x - 38 + 5/x + 6/x^2 = 0$$
$$6(x^2 + 1/x^2) + 5(x + 1/x) - 38 = 0$$

Put $(x + 1/x) = y \Leftrightarrow (x^2 + 1/x^2) = y^2 - 2$

Therefore, equation (1) becomes,

$6(y^2-2)+5y-38=0$
$6y^2-12+5y-38=0$
$6y^2+5y-50=0.$
$6y^2+20y-15y-50=0$
$2y(3y+10)-5(3y+10)=0$
$(3y+10)(2y-5)=0$
$y=-10/3,5/2.$

$x+\frac{1}{x}=-\frac{10}{3}$, $\quad x+1/x=5/2$

$\frac{x^2+1}{x}=\frac{-10}{3}$, $\quad \frac{x^2+1}{x}=\frac{5}{2}$

$3x^2+3=-10x$, $\quad 2x^2+2-5x=0.$
$3x^2+10x+3=0$

$2x^2+4x-x+2=0$, $\quad 3x^2+10x+3=0$

2x (x – 2) -1 (x -2) = 0, $\quad 3x^2+9x+x+3=0$

(2x – 1) (x – 2) = 0, $\quad$ 3x (x + 3) +1 (x + 3) = 0

(3x +1) (x + 3) = 0

X = ½,2 $\quad$ x = - 1/3, -3

The roots of f(x) are 1, 2, ½, - 1/3, -3

To increase or decrease the roots of a given equation by given quantity:

Let $\alpha_1, \alpha_2 \dots \alpha_n$ etc are be the roots of n^{th} degree equation $f(x)=0$.

To form an equation whose roots are decreased by h..

That is $\alpha_1-h, \alpha_2-h, \dots \alpha_n-h$

Let $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$.

Let $y=\alpha_1-h, \alpha_2-h \dots \alpha_{n-h}$

Now, $y=\alpha_1-h \Rightarrow y=x-h(\because x=\alpha_1$ is a root)

$$\Rightarrow x=y+h.$$

Similarly proceeding like this, we get.

$a_0(y+h)^n + (a_1)(y+h)^{n-1} \ldots + a_n = 0$ ……… (1)

Expanding and collecting the coefficients of the powers of y, we get

$A_0 y^n + A_1 y^{n-1} + \cdots A_n = 0$ ……………. (2)

where, $A_0, A_1, A_2 \ldots A_n$ are the functions of $a_0, a_1 \ldots a_n$.

Now, $y = x - h$

$\Rightarrow A_0(x-n)^n + A_1(x-h)^{n-1} + \cdots A_n = 0$ ………. (3)

Equating (1) & (3) as they are identical.

we can easily find the co-efficient $A_0, A_1 \ldots A_n$ By dividing it by $(x-h)^n$

Remainder $= A_0$

Quotient $= A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + \cdots$

Result :-

To form an equation whose roots are decreased by h, we have to divide by h.

To form an equation whose roots are increased by h, we have to divide by $-h$.

Example 1:

Find the Quotient and the Remainder when $3x^3 + 8x^2 + 8x + 12$ is divided by $x - 4$.

Solution:

Given the equation is $3x^3 + 8x^2 + 8x + 2$

To divide the equation by $x - 4$, it is enough to do

$$x - 4 = 0,\ x = 4.$$

	3	8	8	12
4	0	12	80	352
	3	20	88	364

R = 364

Q = $3x^2 + 20x + 88$

Find the Quotient and remainder when $2x^6 + 3x^5 - 15x^2 + 2x - 4$ is by x +5

Solution:

The Given equation is $2x^6 + 3x^5 - 15x^2 + 2x - 4 = 0$

To divide the equation by x+5 it it enough to do

x + 5 = 0, x = - 5

	2	3	0	0	-15	2	-4
-5	0	-10	35	75	875	-4300	21490
	2	-7	35	-175	860	-4298	21486

Q = $2x^5 - 7x^4 + 35x^3 - 175x^2 - 860x - 4298$

R = 21486

Example 3:

To diminish the roots of the equation $x^4 - 5x^3 + 7x^2 - 4x - 5$ by 2

Solution:

Given equation is $x^4 - 5x^3 + 7x^2 - 4x - 5$

	1	-5	7	-4	5
2	0	2	-6	2	-4
	1	-3	1	-2	1
2	0	2	-2	-2	
	1	-1	-1	-4	
2	0	2	2		
	1	1	1		
2	0	2			
	1	3			

The transformed equation is $x^4 + 3x^3 + x^2 - 4x + 1 = 0$

Example 4:

To diminish by 3 the roots of the equation $x^5 - 4x^4 + 3x^3 - 4x - 6 = 0$

Solution:

Given equation is $x^4 - 5x^3 + 7x^2 - 4x - 5$

	1	-4	3	0	-4	6
2	0	3	-3	0	0	-12
	1	-1	0	0	-4	-6
2	0	3	6	18	54	
	1	2	6	18	50	
2	0	3	15	63		

	1	5	21	81
2	0	3	24	
	1	8	45	
	0	3		
	1	11		

The transformed equation is $x^5 + 11x^4 + 45x^3 + 81x^2 + 50x + 6 = 0$.

Example 5:

Find the equation whose roots are the roots of $x^4 - 5x^3 + 7x^2 - 17x + 11$ each diminished by 2.

Solution:

Given equation is $x^4 - 5x^3 + 7x^2 - 17x - 11 = 0$

	1	-5	7	-17	11
2	0	2	-6	2	-30
	1	-3	1	-15	-19
2	0	2	-2	-2	
	1	-1	-1	-17	
2	0	2	2		
	1	1	1		
2	0	2			
	1	3			

Example 6:

Increase by 7 the roots of the equation $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$

Solution:

Increasing the roots by 7 is equivalent to diminishing the roots by -7.

The given equation is $3x^4 + 7x^3 - 15x^2 + x - 2 = 0$

	3	7	-15	1	-2
-7	0	-21	98	-581	4060
	3	-14	83	-580	4058
-7	0	-21	245	-2296	
	3	-35	328	-2876	
-7	0	-21	392		
	3	-56	720		
-7	0	-21			
	3	-77			

The transformed equation is $3x^4 - 77x^3 + 720x^2 - 2876x + 4058 = 0$

Example 7:

Show that the $x^4 - 3x^3 + 4x^2 - 2x + 1 = 0$eqn can be transformed into a reciprocal eqn by diminishing its roots by unity. Hence solve the eqn

Solution:

First to diminish the given equation by 1.

	1	-3	4	-2	1
1	0	1	-2	2	0
	1	-2	2	0	1
1	0	1	-1	1	
	1	-1	1	1	
1	0	1	0		
	1	0	1		
1	0	1			
	1	1			

The transformed equation is $x^4 + x^3 + x^2 + x + 1$. which is a R.E of even degree and like sign.

$x^4 + x^3 + x^2 + x + 1 = 0$ => divide $by 2$

$$x^2 + x + 1 + 1/x + 1/x^2 = 0$$
$$(x^2 + 1/x^2) + (x + 1/x) + 1 = 0$$

Put $y = (x + 1/x)$ & $y^2 - 2 = (x^2 + 1/x^2)$

$$y^2 - 2 + y + 1 = 0$$
$$y^2 + y - 1 = 0$$

$$= \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$-1 + \sqrt{5}/2, -1 - \sqrt{5}/2$$

$x + 1/x = -1 + \sqrt{5}/2 \quad x + 1/x = -1 - \sqrt{5}/2.$
$2(x^2 + 1) = -x + \sqrt{5}x \quad 2x^2 + 2 = -x - \sqrt{5}x.$
$2x^2 + 2 = -x + \sqrt{5}x. \quad 2x^2 + 2 - x(-1 - \sqrt{5})$

$$2x^2 + x - \sqrt{5}x + 2 = 0$$
$$2x^2 - x(-1 + \sqrt{5}) + 2 = 0$$
$$x = -\frac{1 + \sqrt{5} \pm \sqrt{(-1 + \sqrt{5})^2 - 16}}{4}$$
$$x = \frac{\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}}}{4} \text{ and } x = \frac{-(\sqrt{5} + 1) + \sqrt{-10 + 2\sqrt{5}}}{4}$$

The Roots of the original equation are the above roots increased by 1,

$$x = \frac{\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}}}{4} + 1, \qquad x = \frac{-(\sqrt{5} + 1) \pm \sqrt{-10 + 2\sqrt{5}}}{4}$$
$$x = \frac{\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}} + 4}{4}, \qquad x = \frac{-(\sqrt{5} + 1) \pm \sqrt{-10 + 2\sqrt{5}} + 4}{4}$$
$$x = \frac{\sqrt{5} + 3 \pm \sqrt{-10 - 2\sqrt{5}}}{4}, \qquad x = \frac{-\sqrt{5} + 3 \pm \sqrt{-10 + 2\sqrt{5}}}{4}$$

Exercises 1:

1. Find the equation whose roots are the roots of $x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$ each diminished by 2.
2. Find the equation whose roots are the roots of $4x^5 - 2x^3 + 7x - 3 = 0$ each increased by 2.
3. Find the equation each of whose roots exceeds by 2 a root of the equation $x^3 - 4x^2 + 3x - 1 = 0$

Horner's Method:

Procedure :-

Horner's method is used to determine a real root of a numerical polynomial equation $f(x) = 0$, correct to given place of decimal (ie) The root is $a \cdot \alpha_1 d_2 d_3$ ....

Step 1:-

We are going to find the integral part a by trial find 2 consecutive integers where a real positive roots of the given equation lies.

Let a and $a+1$ be & consecutive integers, such that $f(a)$ and $f(a+1)$ are opposite sign, therefore a root lies between a and $a+1$.

Therefore the integral part of root is a. Let the root be $a.d_1, d_2, d_3$ ...

Step 2:-

To find d_1

To diminish the roots of the equation by a. Now, equation is

$\phi_1(x) = 0$ .... (1)

will have roots between zero and one. multiply the roots of (1) by 10 .

(i.e.) The coefficients of $\phi_1(x)$ are multiplied by $1,10,100,1000,$ ... respectively. by trial find the integer between which the roots of (1) lies which is d_1 Now equation will be $\phi_2(x) = 0$.

Example 1:

The equation $x^3 - 3x + 1 = 0$. has a root between 1 and 2. Calculate it to three places of decimal.

Solution:

Given equation is $x^3 - 3x + 1 = 0$

Let $f(x) = x^3 - 3x + 1$

$f(1) = 1 - 3 + 1 = -1 \quad (-ve)$

$f(2) = 8 - 6 + 1 = 3$ (tie)

The root is $1.d_1d_2d_3$.

step 1:- To find d_1

	1	0	-3	1
1	0	1	1	-2
	1	1	-2	-1
1	0	1	2	
	0	2	0	
1	0	1		
	0	3		

The transformed equation is $x^3 + 3x^2 - 1 = 0$. Let $f_1(x) = x^3 + 3x^2 - 10$

Multiply the roots of the transformed equation $f_1(x)$ by 10

That is (i.e.) Multiply by 1,10,100,1000 . ... etc to the coefficients of x^3, x^2, x and constant term respectively.

Therefore the transformed equation is $f_2(x) = x^3 + 304^2 - 1000 = 0$

Now,

$$f_2(1) = 1 + 30 - 1000 < 0$$
$$f_2(2) = 8 + 120 - 1000 < 0$$
$$f_2(3) = 27 + 100 - 1000 < 0$$
$$f_2(4) = 64 + 480 - 1000 < 0$$
$$f_2(5) = 125 + 750 - 1000 < 0$$
$$f_2(6) = 216 + 1080 - 1000 > 0$$

∴ The roots of $f_2(x)$ lies between 5 and 6

step 2:-

Let the root be $1.5d_2d_3$.

To find d_2, diminish the transformed equation by 5

	1	30	0	-1000
5	0	5	175	875
	1	35	175	-125
5	0	5	200	
	1	40	375	
5	0	5		
	1	45		

The transformed equation is $x^3 + 45x^2 + 375x - 125 = 0$.

$$\text{Let } f_3(x) = x^3 + 45x^2 + 375x - 125 = 0.$$

Multiply the roots of the transformed equation $f_3(x)$ by 10

That is (i.e.) Multiply by $1{,}10{,}100{,}1000$ etc to the coefficients of x^3, x^2, x and constant term respectively.

The transformed equation is $f + (x) = x^3 + 450x^2 + 37500x - 125000$.

$$f_4(0) = -125000 < 0$$
$$f_{4(1)} = -ve.$$
$$f_4(2) = -ve.$$
$$f_4(3) = -ve$$
$$f_4(4) = \text{ the}$$

The root lies between 3& 4 .

let the root be 1.53 d_3

Step 3 :

To find d_3, diminish the root by 3 .

	1	450	37500	-125000
3	0	3	1359	116577
	1	453	38859	-8423
3	0	3	1368	
	1	456	40227	
3	0	3		
	1	459		

The transformed equation is $x^3 + 459x^2 + 40227\, x - 8423 = 0$

Let $f_5(x) = x^3 + 459x^2 + 40227x - 8423$

Multiply the roots of the transformed equation $f_5(x)$ by 10

That is (i.e.) Multiply by 1,10,100, 1000 etc to the coefficience of x^3, x^2 and x & constant term respectively.

the transformed equation is $f_5(x) =$

$$x^3 + 4590x^2 + 4022700\,\mathrm{x} - 8423000$$
$$f_6(0) = -8423000 < 0$$
$$f_6(1) =< 0$$
$$f_6(2) =< 0$$
$$f_6(3) => 0$$

The root lies between 2 and 3 .

The root is 1.532 .

Example 2:

Find the positive root of the equation $x^3 - 2x^2 - 3x - 4 = 0$ correct to 3 places of decimals.

Solution:

Given equation is $x^3 - 2x^2 - 3x - 4 = 0$

Let $f(x) = x^3 - 2x^2 - 3x - 4$

$$f(1) = 1 - 2 - 3 - 4 = -8 < 0.$$
$$f(2) = 8 - 8 - 6 - 4 = -10 < 0$$
$$f(3) = 27 - 18 - 9 - 4 = -4 < 0$$
$$f(4) = 64 - 32 - 12 - 4 => 0.$$

The root lies between 3 & 4 .

Let the roots be $3 \cdot d_1 d_2 d_3$.

Step 1:

To find d_1, diminish by 3 .

	1	-2	-3	-4
3	0	3	3	0
	1	1	0	-4
3	0	3	12	
	1	4	12	
3	0	3		
	1	7		

The transformed equation is,

$$x^3 + 7x^2 + 12x - 4 = 0$$

Let $f_1(x) = x^3 + 7x^2 + 12x - 4 = 0.$

Multiply the roots of the transformed equation $f_1(x)$ by 10

That is (i.e) Multiply by 1,10,100,1000 ... etc to the coefficience of x^3, x^2, x and constant term respectively.

Therefore the transformed equation is

$$f_2(x) = x^3 + 70x^2 + 1200x - 4000 = 0.$$

Now, $f_2|2| =< 0$

$f_2 2(2) =< 0$

$f_2(3) => 0$

The root lies between 2 and 3 .

The root is $3.2d_2\ d_3$

To find d_2.

	1	70	1200	-4000
2	0	2	144	2688
	1	72	1344	-1312
2	0	2	1480	
	1	74	1492	
2	0	2		
	1	76		

$$f_3(x) = x^3 + 76x^2 + 1492x - 1312$$

Multiply the roots of the transformed equation $f_3(x)$ by 10

That is (i.e.) Multiply by 10,100,1000 …. etc for x^3, x^2, x and constant term respectively

$$\text{Let } f_4(x) = x^3 + 760x^2 + 149200x - 1312000$$

$$f_4(0) = -1312000 < 0$$

$$f_4(1) = 1 + 760 + 149200 - 1312000 < 0$$

$$f_4(2) = 8 + 3040 + 298400 - 1312000 < 0$$

$$f_4(3) = 27 + 684 + 447600 - 1312000 < 0$$

$$f_4(4) = 64 + 12{,}160 + 596800 - 1312090 < 0$$

$$f_4(5) = 125 + 19000 + 746000 - 1312000 < 0$$

$$f_4(6) = < 0$$

$$f_4(7) = < 10$$

$$f_4(8) = < 20$$

$$f_4(9) = > 0$$

The root lies between 8 and 9 .

The root is $3 \cdot 28d_3$

To find d_3.

To find d_3, diminish the roots by 8 .

	1	760	149200	-13,12,000
8	0	8	6144	12,42,752
	1	768	155344	-69248
8	0	8	6208	
	1	776	161552	
8	0	8		
	1	784		

$x^3 + 784x^2 + 161552x - 69248.$

let $f_5(x) = x^3 + 784x^2 + 161552x - 69248$

Multiply the roots of the transformed equation $f_5(x)$ by 10

That is (i.e.) Multiply by 10,100,1000. for $x^3 - x^2, x$ and constant term respectively.

$$\text{let } f_6(x) = x^3 + 7840x^2 + 16155200x - 69248000.$$

$$f_6(0) = -ve$$
$$f_6(1) = -ve$$
$$f_6(2) = -ve$$
$$f_6(3) = -ve$$
$$f_6(4) = -ve$$
$$f_6(5) = +\text{ve}$$

The root lies between 4 and 5

The root is 3.284 .2617

Example 3:

Find the root between 0 and 1 correct to 3 places of decimal $x^3 + 18x - 6 = 0.$

Solution:

Given: $x^3 + 18x - 6 = 0.$

Let $f(x) = x^3 + 18x - 6 = 0$

$f(0) = (-\text{ ve })$
$f(1) = (\text{ +ve })$

The rot lies between 0 & 1

The root is $0 \cdot d_1 d_2 d_3.$

step 1:- To find d_1, diminish by 0 .

	1	0	18	-6
0	0	0	0	0
	1	0	18	-6
0	0	0	0	
	1	0	18	
0	0	0		
	1	0		

The transformed equation is $x^3 + 18x - 6 - 0$.

Let $f_1(x) = x^3 + 18x - 6 = 0$.

Multiply the roots of the transformed equation by 10.

That is (i.e.) Multiply by 1,10,100, 1000 ... etc to the coefficience of x^3, x^2, x and constant term respectively.

Therefore the transformed eqn

$$f_2(x) = x^3 + 1800 \times -6000$$
$$f_2(0) = (-ve)$$
$$f_2(1) = (-ve)$$
$$f_2(2) = (-ve)$$
$$f_2(3) = (-ve)$$
$$f_2(4) = (+ve)$$

The root is $0.3d_2d_3$

diminish the eqn $f_2(x)$ by 3 .

	1	0	1800	-6000
3	0	3	9	5427
	1	3	1809	-573
3	0	3	18	
	1	6	1827	
3	0	3		
	1	9		

transformed equation is

$f_3(x) = x^3 + 9x^2 + 1827x - 573$

Multiply the roots of the transformed equation by 10.

Multiply by 1, 10, 100 and 1000 ... for x^3, x^2, x and constant term respectively The transformed equation is

$$f_4(x) = x^3 + 90x^2 + 182700x - 573000.$$
$$f_4(0) = (-v_e)$$
$$f_4(1) = (-v_e)$$
$$f_4(2) = (-v_e)$$
$$f_4(3) = (-v_e)$$
$$f_4(4) = (v_e)$$

The Root lies between 3 and 4

The Root is 0.33 d_3.

To find d_3, diminish the root by 3 .

	1	90	182700	-573000
3	0	3	279	548937
	1	93	182979	-24063
3	0	3	288	
	1	96	183267	
3	0	3		
	1	99		

$f_5(x) = x^3 + 99x^2 + 183267x - 24063.$

Multiply the roots of the equation $f_5(x)$ by 10

Multiply by 1,10,100,1000 ... etc for x^3, x^2, x and constant term respectively.

The eq $f_6(x) = x^3 + 990x^2 + 18326700$ -24063000 .

$$f_6(0) = (\text{-ve})$$
$$f_6(1) = (-ve)$$
$$f_6(2) = (\text{+ve})$$

The Root lies between 1 and 2

The Root is 0.331.

Exercises:

1. Find the root between 2 and 3 correct to two places of decimal $x^3 - 5x - 11 = 0$.
2. Find the real root $x^3 + 6x = 2$ to three places of decimals.
3. Find the real root $x^3 - 3 = 0$ to three places of decimals.

UNIT II : SUMMATION OF SERIES

Summation of series using Binomial, Exponential and Logarithmic series.

BINOMIAL SERIES

When n is a positive integer $(x + a)^n$ can be expanded as $(x+a)^n = x^n + {}_nC_1 . x^{n-1}a + {}_nC_2 . x^{n-2}a^2 + \ldots + {}_nC_r . x^{n-r} . a^r + \ldots + a_n$.. This is known as the binomial theorem for the positive integer n. When n is a rational number $(1 + x)^n$ can be expanded as an infinite series when $-1 < x < 1$ (i.e) $|x| < 1$ and it is given by

$$(1 + x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (1)$$

This is known as binomial series for $(1+ x)^n$ where n is a rational number.

General term

The $(r + 1)^{th}$ term in the expansion is often denoted by

U_{r+1} or T_{r+1} . $U_{r+1} = {}_nC_r \, x^{n-r} \, a^r$

We may obtain any particular term by giving r particular values. Thus the first term is obtained by writing $r = 0$, the second by writing r = 1 and so on . So the $(r +1)^{th}$ term is called the general term.

Thus we get $(x+a)^n = \sum_{r=0}^{n} {}_nC_r x^{n-r} a^r$

Note:-

(1) The expansion contains (n + 1) terms.

(2) The numbers ${}_nC_0, {}_nC_1 \ldots {}_nC_r \ldots {}_nC_n$ are called the Binomial Coefficients. They are sometimes written as C_0 , C_1 , C_n.These binomial coefficients are all integers since ${}_nC_r$ is the number of combinations of *n* things taken *r* at a time.

(3) Since $C_0 = C_n$, $C_1 = C_{n-1}$, …… $C_r = C_{n-r}$, the coefficients of terms equidistant from the beginning and the end of the expansion are equal.

Summation of various series involving Binomial Coefficients

It is convenient to write the Binomial theorem in the form

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_r x^r + \ldots C_n x^n$.

We can seen in the expansion that the coefficients of terms which are equidistant from the beginning and the end are equal.

$\therefore$ $C_0 = C_n = 1$, $C_1 = C_{n-1} = n$ …. and in general.

$$C_r = C_{n-r} = \frac{n!}{r!(n-r)!}.$$

Some important particular cases of the Binomial expansion.

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots..$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots..$$

$$(1-x)^{-3} = \frac{1}{2}\{1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \ldots.\}$$

$$(1-x)^{-4} = \frac{1}{6}\{1.2.3 + 2.3.4x + 3.4.5x^2 + 4.5.6x^3 + \ldots\}$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots\ldots$$

$$(1-x)^{-1/3} = 1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots\ldots$$

Application of the Binomial theorem to the summation of series.

We have proved when $|x| < 1$, for all values of n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

Solved problems

Example 1. Find the sum to infinity of the series $1+\frac{3}{4}+\frac{3}{4}.\frac{5}{8}+\frac{3}{4}.\frac{5}{8}+\frac{7}{12}+\ldots$

Solution.

The factors in the numerators form an A.P with common difference 2: we therefore divide each of these by 2.

Each of the factors in the denominator has 4 for a factor; removing 4 from each will leave a factorial . Hence we have

$$1+\frac{\frac{3}{2}}{1}.\frac{2}{4}+\frac{\frac{3}{2}.\frac{5}{2}}{1.\ 2}.\left(\frac{2}{4}\right)^2+\frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}{1.\ 2.\ 3}.\left(\frac{2}{4}\right)^3+\ldots\ldots$$

i.e., $1+\frac{\frac{3}{2}}{1!}.\frac{1}{2}+\frac{\frac{3}{2}.\frac{5}{2}}{2!}.\left(\frac{2}{4}\right)^2+\frac{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}{3!}.\left(\frac{2}{4}\right)^3+\ldots\ldots$

Put $n=\frac{3}{2}$ and $x=\frac{1}{2}$.

Then the series becomes

$$1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots\ldots$$

$$=(1-x)^{-n}$$

$$=(1-\tfrac{1}{2})^{-3/2}$$

$$=2\sqrt{2}.$$

Example 2. Sum the series to infinity $\frac{1\ .\ 4}{5\ .\ 10}-\frac{1\ .\ 4\ .\ 7}{5\ .\ 10\ .\ 15}+\frac{1\ .\ 4\ .\ 7\ .\ 10}{5\ .\ 10\ .\ 15\ .\ 20}\ldots..$

Solution.

The numerators form an A.P . with 3 as common difference and the denominators are factorials, each of whose factors has been multiplied by 5.

$\therefore$ The series can be written as

$$S = \frac{\frac{1}{3}\cdot\frac{4}{3}}{1\,.\,2}\,.\left(-\frac{3}{5}\right)^2 + \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{1\,.\,2\,.\,3}\,.\left(-\frac{3}{5}\right)^3 + \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdot\frac{10}{3}}{1\,.\,2\,.\,3\,.\,4}\,.\left(-\frac{3}{5}\right)^4 + \ldots.$$

Put $n = \frac{1}{3}$ and $x = -\frac{3}{5}$.

$$\therefore\ S = \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \frac{n(n+1)(n+2)(n+3)}{4!}x^4 \ldots..$$

$$= 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3$$

$$= (1-x)^{-n} - 1 - nx$$

$$= (1+\tfrac{3}{5})^{-1/3} - 1 + \frac{1}{3}\,.\,\frac{3}{5} = \frac{1}{2}\,(5)^{1/3} - \frac{4}{5}.$$

Example 3. Sum the series to infinity. $\frac{15}{16} + \frac{15.21}{16.24} + \frac{15.21.27}{16.24.32} + \ldots\ldots$

Solution.

The factors in the numerator form an A.P. with common difference 6 and those of the denominator an A.P with common difference 8.

Let S be the sum of the series.

$$\text{Then } S = \frac{\frac{15}{6}}{2}.\left(\frac{6}{8}\right) + \frac{\frac{15}{6}\cdot\frac{21}{6}}{2\,.3}.\left(\frac{6}{8}\right)^2 + \frac{\frac{15}{6}\cdot\frac{21}{6}\cdot\frac{27}{6}}{2.3.4}.\left(\frac{6}{8}\right)^3 + \ldots..$$

The factors of the denominators do not begin with 1. Hence one additional factor , namely unity, has to be introduced into the denominator of each coefficient. The number of factors in the numerator is to be the same as that of the factors in the denominator. So we have to introduce an additional factor in the numerator also, which factor is clearly $\frac{9}{6}$.

$$\therefore\ \frac{9}{6}S = \frac{\frac{9}{6}\cdot\frac{15}{6}}{1.2}\left(\frac{6}{8}\right) + \frac{\frac{9}{6}\cdot\frac{15}{6}\cdot\frac{21}{6}}{1.2.3}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}\cdot\frac{15}{6}\cdot\frac{21}{6}\cdot\frac{27}{6}}{1.2.3.4}\left(\frac{6}{8}\right)^3 + \ldots$$

Since the index of x in every term must be the same as the number of factors in the numerator or denominator of the coefficient, we have

$$S.\,\frac{9}{6}\,.\,\frac{6}{8} = \frac{\frac{9}{6}\cdot\frac{15}{6}}{2!}\left(\frac{6}{8}\right)^2 + \frac{\frac{9}{6}\cdot\frac{15}{6}\cdot\frac{21}{6}}{3!}\left(\frac{6}{8}\right)^3 + \ldots$$

-

Put $\frac{9}{6} = \text{n}$ and $\text{x} = \frac{6}{8}$.

$$\therefore \frac{9}{6}S = \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots$$

$$= 1 + \frac{n}{1!} + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots - (1+nx)$$

$$= (1-x)^{-n} - (1+nx)$$

$$= (1-\tfrac{6}{8})^{-9/6} - (1+\tfrac{9}{6}.\tfrac{6}{8})$$

$$= (\tfrac{1}{4})^{-3/2} - (1+\tfrac{9}{8})$$

$$= \frac{47}{8}.$$

$$\therefore \quad S = \frac{47}{9}.$$

Example 4. Find the sum of to infinity of the series $\frac{1}{24} - \frac{1.3}{24.32} + \frac{1.3.5}{24.32.40} - \ldots.$

Solution.

Proceeding as in the previous example, we get

$$S = \frac{\frac{1}{2}}{3}.\left(\frac{2}{8}\right) + \frac{\frac{1}{2}.\frac{3}{2}}{3.4}.\left(\frac{2}{8}\right)^2 + \frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{3.4.5}.\left(\frac{2}{8}\right)^3 + \ldots..$$

In order to express this in the standard binomial form, the factor 1 . 2 must be inserted in each denominator, and two additional factors must be then inserted in each numerator to secure that the number of factors in the numerator is the same as that in the denominator. In order that the factors of the numerator may remain in A.P. the additional factors(which should be the same in each term) must be $-\frac{3}{2}, \frac{1}{2}$.

$$\therefore \quad -\frac{3}{2}.-\frac{1}{2}.S.\frac{1}{1.2} = \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}}{1.23}.\left(\frac{2}{8}\right) - \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{1.2.3.4}.\left(\frac{2}{8}\right)^2 + \frac{-\frac{3}{2}.-\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{1.2.3.4.5}.\left(\frac{2}{8}\right)^3$$

The index of x should be the same as the number of factors in the numerator.

$\therefore$ The series is to be multiplied by $\left(\frac{2}{8}\right)^2$.

$$\therefore\ -\frac{3}{2}.-\frac{1}{2}.S.\ \frac{1}{2}.\left(\frac{2}{8}\right)^2$$

$$=\frac{-\frac{3}{2}.-\frac{1}{2}\frac{1}{2}}{3!}\left(\frac{2}{8}\right)^3-\frac{-\frac{3}{2}.-\frac{1}{2}\frac{3}{2}\frac{5}{2}}{4!}.\left(\frac{2}{8}\right)^4+\frac{-\frac{3}{2}.-\frac{1}{2}\frac{1}{2}\frac{3}{2}\frac{5}{2}}{5!}.\left(\frac{2}{8}\right)^5+\ldots\ldots$$

i.e., $\frac{3S}{128}=\frac{n(n+1)(n+2)}{3!}x^3-\frac{n(n+1)(n+2)(n+3)}{4!}x^4+\ldots\ldots$

If $n=-\frac{3}{2}$, $x=\frac{2}{8}$.

$$\therefore\ \frac{3S}{128}=-(1+x)^{-n}+\{1-nx+\frac{n(n+1)}{2!}x^2\}$$

$$=-(1+\frac{2}{8})^{3/2}+\{1+\frac{3}{2}.\frac{2}{8}+\frac{-\frac{3}{2}.-\frac{1}{2}}{2!}\left(\frac{2}{8}\right)^2\}$$

$$=\frac{-5\sqrt{5}}{8}+1+\frac{3}{8}+\frac{3}{128}$$

$$=\frac{179}{128}-\frac{-5\sqrt{5}}{8}.$$

$$\therefore\ S=\frac{1}{3}(179-80\sqrt{5}).$$

Exercises

Find the sum to infinity of the following series:

(1) $\frac{3}{1}+\frac{3.5}{1.2}.\frac{1}{3}+\frac{3.5.7}{4.8.12}+\ldots\ldots$

(2) $\frac{3}{50}+\frac{3.18}{50.100}+\frac{3.18.33}{50.100.150}+\ldots\ldots$

(3) $\frac{5}{3.6}+\frac{5.7}{3.6.9}+\frac{5.7.9}{3.6.9.12}+\ldots\ldots$

(4) $\frac{3}{18}+\frac{3.7}{18.24}+\frac{3.7.11}{18.24.30}+\ldots\ldots$

(5) $\frac{5}{3.6}.\frac{1}{4^2}+\frac{5.8}{3.6.9}.\frac{1}{4^3}+\frac{5.8.11}{3.6.9.12}.\frac{1}{4^4}+\ldots.$

(6) $\frac{1}{2^3(3!)}-\frac{1.3}{2^4(4!)}+\frac{1.3.5}{2^5(5!)}+\ldots\ldots$

Approximate values.

The Binomial series can be used to obtain approximate values and limits of expressions as follows.

Example 1. Find correct to six places of decimals the values of $\frac{1}{(9998)^{1/4}}$.

Solution.

$$\left(\frac{1}{(9998)}\right)^{1/4} = \frac{1}{(10000-2)^{1/4}}$$

$$= \frac{1}{(10^4-2)^{1/4}}$$

$$= \frac{1}{10(1-\frac{2}{10^4})^{1/4}}$$

$$= \frac{(1-\frac{2}{10^4})^{-1/4}}{10}$$

$$= \frac{1+\frac{1}{4}\cdot\frac{2}{10^4}+\frac{\frac{1}{4}\cdot\frac{5}{4}}{2!}\cdot\frac{4}{10^8}+\cdots}{10}$$

$$=\frac{1}{10}+\frac{1}{2}\cdot\frac{1}{10^5}+\frac{5}{8}\cdot\frac{1}{10^9}+\ldots.$$

$$= 0.1 + \frac{1}{2}(0.00001) + \frac{5}{8}(0.000000001)$$

$$= 0.1 + 0.000005 + 0.0000000005$$

$$= 0.1000050005$$

$\therefore\ \frac{1}{(9998)^{1/4}} = 0.100005$ correct to six places of decimals.

Example 2. Calculate correct to six places of decimals $(1.01)^{1/2} - (0.99)^{1/2}$.

Solution.

Write x = 0.01.

$$\therefore\ (1.01)^{1/2} = (1+x)^{1/2}$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}.(-\frac{1}{2}).(-\frac{3}{2})}{3!}x^3 + \ldots.$$

$$(0.99)^{1/2} = (1-x)^{1/2}$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}.(-\frac{1}{2})}{2!}x^2 - \frac{\frac{1}{2}.(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \ldots.$$

$$\therefore (1.01)^{1/2} - (0.99)^{1/2} = 2\left\{\frac{1}{2}x + \frac{\frac{1}{2}.(-\frac{1}{2}).(-\frac{3}{2})}{3!}x^3 + \frac{\frac{1}{2}.(-\frac{1}{2}).(-\frac{3}{2}).(-\frac{5}{2})(-\frac{7}{2})}{3!}x^5 + \cdots\right\}$$

$$= 2\left\{\frac{1}{2}x + \frac{1}{16}x^3 + \frac{7}{256}x^5 + \cdots\right\}$$

$$= x + \frac{1}{8}x^3 + \frac{7}{128}x^5 + \cdots$$

$$= (0.01) + \frac{1}{8}(0.01)^3 + \frac{7}{128}(0.01)^5 + \cdots$$

$= 0.01 + \frac{1}{8}(0.000001)$ + terms not affecting the 8^{th} decimal place

$$= 0.01 + 0.000000125$$

$= 0.010000125$

$\therefore (1.01)^{1/2} - (0.99)^{1/2} = 0.010000$ correct to six places of decimals.

Exercises

1. Find the value of $\frac{1}{(128)^{1/3}}$ correct to five places of decimals.
2. Find the expansion of $(1+\frac{1}{64})^{1/3}$ and find the cube root of 65 correct to three places of decimals.
3. Prove that $(2)^{1/3} = 1\frac{1}{4}(1+0.024)^{1/3}$ and hence find the cube root of two to four places of decimals.
4. Evaluate $\left(\frac{0.998}{1.002}\right)^{1/3}$ correct to four places of decimals, without using logarithms.
5. Find to five places of decimals the value of $(1003)^{\frac{1}{3}} - (997)^{\frac{1}{3}}$.

Answers: 1. 0.19842, 2. 4.021, 4. 1.0027, 5. 0.02000.

Example 1. When x is small, prove that $\frac{(1-3x)^{-2/3}+(1-4x)^{-3/4}}{(1-3x)^{-1/3}+(1-4x)^{-1/4}} = 1+\frac{3}{2}x + 4x^2$ approximately.

Solution.

The expression is equal to

$$= \frac{1+\frac{2}{3}.3x+\frac{\frac{2}{3}\cdot\frac{5}{3}}{2!}(3x)^2+\frac{\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}}{3!}(3x)^3+\cdots+1+\frac{3}{4}.4x+\frac{\frac{3}{4}\cdot\frac{7}{4}}{2!}(4x)^2+\cdots}{1+\frac{1}{3}.3x+\frac{\frac{1}{3}\cdot\frac{4}{3}}{2!}(3x)^2+\frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{3!}(3x)^3+\cdots+1+\frac{1}{4}.4x+\frac{\frac{1}{4}\cdot\frac{5}{4}}{2!}(4x)^2+\cdots}$$

Since x^3 and higher powers of x may be neglected the expression

$$= \frac{2+5x+15\frac{1}{2}x^2}{2+2x+4\frac{1}{2}x^2}$$

$$= \frac{(2+5x+15\frac{1}{2}x^2)}{2(1+x+\frac{9}{4}x^2)}$$

$$= \frac{2+5x+15\frac{1}{2}x^2}{2} \cdot (1+x+\frac{9}{4}x^2)^{-1}$$

$$= (1+\frac{5}{2}x+\frac{31}{4}x^2)\{1+x(1+\frac{9}{4}x)\}^{-1}$$

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\{1-x\left(1+\frac{9}{4}x\right)+x^2(1+\frac{9}{4}x)^2 \ldots\}$$

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left(1-x-\frac{9}{4}x^2+x^2\right)$$

(x^3 and higher powers of x neglected)

$$= \left(1+\frac{5}{2}x+\frac{31}{4}x^2\right)\left(1-x-\frac{5}{4}x^2\right)$$

$$= 1+\frac{5}{2}x+\frac{31}{4}x^2-x-\frac{5}{2}x^2-\frac{5}{4}x^2$$

$$= 1+\frac{3}{2}x+4x^2.$$

Example 2. Show that $\sqrt{x^2+16}-\sqrt{x^2+9} = \frac{7}{2x}$ nearly for sufficiently large values of x.

Solution.

The expression $= (x^2+16)^{1/2}-(x^2+9)^{1/2}$

$$= x(1+\frac{16}{x^2})^{1/2}-x(1+\frac{9}{x^2})^{1/2}$$

$$= x(1+\frac{1}{2}.\frac{16}{x^2}-\cdots)-x(1+\frac{1}{2}.\frac{9}{x^2}-\cdots)$$

(Since $\frac{1}{x}$ is small, the expansion is valid)

$$= x+\frac{8}{x}-x-\frac{9}{2x}$$

$$= \frac{7}{2x} \text{ nearly.}$$

Exercises

1. If x be so small that its square and higher powers may be neglected, find the value of

   $(1-7x)^{1/3}-(1+2x)^{-3/4}$

2. When x is small , show that $\frac{(1-x)^{-5/2}+(16+8x)^{1/2}}{(1+x)^{-1/2}+(2+x)}= 1+\frac{23}{40}x^2$ approximately.

3. If x be so small that its squares and higher powers may be neglected. Prove that $\frac{(9+2x)^{1/2}+(3+4x)}{(1-x)^{1/3}}=9+\frac{74}{5}x$ nearly.

4. If x be so small that powers of x above x^3 may be neglected, show that $\frac{(1+x+x^2)+(1+x)^2}{(1-x)^{1/3}}= 1+4x+7x^2+6x^3$.

5. If c is small in comparison with *l*, show that $(\frac{l}{l+c})^{1/2}+(\frac{l}{l-c})^{1/2}=2+\frac{3c^3}{4l^2}$ approximately.

6. Show that $\sqrt{x^2+4}-\sqrt{x^2+1}$ is $1-\frac{1}{4}x^2+\frac{7}{64}x^4$ nearly when x is small and $\frac{3}{2x}(1-\frac{3}{4x^2}+\frac{3}{8x^4})$ nearly when x is large.

   Answer: 1.1 $-\frac{23}{6}x$.

$\frac{5S}{72} + \left(1 - \frac{5}{6} + \frac{5}{72}\right) = (1-x)^{-p/q}$ where p = 5;q = 4 and $\frac{x}{q} = \frac{1}{6}$ and hence $x = \frac{2}{3}$

Therefore $\frac{5S}{72} + \frac{17}{72} = \left(1 - \frac{2}{3}\right)^{5/4}$

$$\therefore \frac{5S}{72} = \left(\frac{1}{3}\right)^{5/4} - \frac{17}{72}$$

$$\therefore S = \frac{72}{5}\left[\frac{3^{-5/4}(72)-17}{72}\right] = \frac{72}{5}\left[\frac{3^{-5/4}(3)^2 8-17}{72}\right]$$

$$= \frac{72}{5}\left[\frac{3^{3/4}(8)-17}{72}\right] = \frac{72}{5}\left[\frac{8(27)^{1/4}-17}{72}\right]$$

$$S = \frac{1}{5}\left(8(27)^{1/4} - 17\right).$$

Exponential Series

We will learn some series which can be summed up by exponential series. We have proved that for all real values of x.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + .. \text{ to } \infty \qquad \ldots\ldots\ldots\ldots..(1)$$

In particular when x = 1 , we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + .. \text{ to } \infty \qquad \ldots\ldots\ldots\ldots..(2)$$

and when x = −1 , we have

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n . \frac{1}{n!} + .. \text{ to } \infty \qquad \ldots\ldots\ldots\ldots..(3)$$

Changing x into $-x$ in series (1) , we get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots + (-1)^n . \frac{x^n}{n!} + \cdots \qquad \ldots\ldots\ldots\ldots..(4)$$

Adding (1) and (4) , we get

$$\frac{e^x - e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots\ldots(5)$$

Subtracting (4) from (1) , we get

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(6)$$

When x = 1 , series (5) and (6) become

$$\frac{e+e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(7)$$

$$\frac{e-e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots \text{ to } \infty \qquad \ldots\ldots\ldots\ldots(8)$$

Note. It can be verified that e is an irrational number whose value lies between 2 and 3. Further the value of e correct to four places of decimals is given by e = 2.7183. We shall use these series to find the sums of certain series. The different methods are illustrated by the following worked examples..

Example. Sum the series $1 + \frac{1+3}{2!} + \frac{1+3+3^3}{3!} + \frac{1+3+3^2+3^3}{4!} + \cdots$ to ∞.

Solution.

Let u_n be the n^{th} term of the series and S be the sum to infinity of the series.

$$\therefore \quad u_n = \frac{1+3+3^2+\ldots\ldots\ldots+3^{n-1}}{n!}$$

$$= \frac{3^n-1}{3-1} \cdot \frac{1}{n!}$$

$$= \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

$$\therefore \quad u_1 = \frac{1}{2}\left(\frac{3^1}{1!} - \frac{1}{1!}\right)$$

$$u_2 = \frac{1}{2}\left(\frac{3^2}{2!} - \frac{1}{2!}\right)$$

$$u_3 = \frac{1}{2}\left(\frac{3^3}{3!} - \frac{1}{3!}\right)$$

…………………

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$$u_n = \frac{1}{2}\left(\frac{3^n}{n!} - \frac{1}{n!}\right)$$

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$$S = \frac{1}{2}\left(\frac{3^1}{1!} + \frac{3^2}{2!} + \cdots + \frac{3^n}{n!} + \cdots\right) - \frac{1}{2}\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots\right)$$

$$= \frac{1}{2}(e^3 - 1) - \frac{1}{2}(e - 1)$$

$$= \frac{1}{2}e(e^2 - 1).$$

Exercises

1. Show that $(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots)^2 = (1 + \frac{1}{3!} + \frac{1}{5!} + \cdots)^2$
2. Show that $\frac{e+1}{e-1} = \frac{\frac{1}{1!}+\frac{1}{3!}+\cdots}{\frac{1}{2!}+\frac{1}{4!}+\cdots}$.
3. Show that $2\{1 + \frac{(\log_e n)^2}{2!} + \frac{(\log_e n)^4}{4!} + \ldots\} = \left(n + \frac{1}{n!}\right)$.
4. Show that $\sum_1^\infty \frac{n-1}{n!} = 1$.

If the given series is $\sum_{n=0}^\infty f(n).\frac{x^n}{n!}$ where f (n) is a polynomial in n of degree r , we can find constants $a_0, a_1, \ldots \ldots a_r$ so that

$f(n) = a_0 + a_1 n + a_2 n(n-1) \ldots . + a_r n(n-1) \ldots . (n - r + 1)$ and then

$$\sum_{n=0}^\infty f(n).\frac{x^n}{n!} = a_0 \sum_{n=0}^\infty \frac{x^n}{n!} + a_1 \sum_{n=0}^\infty \frac{x^n}{(n-1)!} + \ldots . + a_r \sum_{n=0}^\infty \frac{x^n}{(n-r)!}$$

$$= a_0 . e^x + a_1 x . e^x + \ldots . . + a_r . x^r . e^x$$

$$= (a_0 + a_1 x + a_2 x^2 + \ldots . a_r x^r)\, e^x$$

Example 1. Sum the series $\sum_{n=0}^\infty \frac{(n+1)^3}{n!}. \; x^n$.

Solution.

Put $(n+1)^3 = A + B\,n + C\,n\,(n-1) + D\,n\,(n-1)(n-2)$.

Putting $n = 0$, 1 , 2 and equating the coefficients of n^3, we get

A = 1, B = 7, C = 6, D = 1.

Let the sum of the series be S.

$$S = \sum_0^\infty \frac{1+7n+6n(n-1)+n(n-1)(n-2)}{n!} \; x^n$$

$$= \sum_0^\infty \frac{x^n}{n!} + 7\sum_0^\infty \frac{x^n}{(n-1)!} + 6\sum_0^\infty \frac{x^n}{(n-2)!} + \sum_0^\infty \frac{x^n}{(n-3)!}$$

Now $\sum_0^\infty \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$

$$\sum_0^\infty \frac{x^n}{(n-1)!} = x + \frac{x^2}{1!} + \frac{x^3}{2!} \ldots = x.e^x$$

$$\sum_0^\infty \frac{x^n}{(n-2)!} = x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} \ldots = x^2.e^x$$

$$\sum_0^\infty \frac{x^n}{(n-3)!} = x^3 + \frac{x^4}{1!} + \frac{x^5}{2!} \ldots = x^3.e^x$$

∴ $S = (1 + 7x + 6x^2 + x^3)\, e^x$.

Example 2. Sum the series $\frac{1^2}{1!} + \frac{1^2+2^2}{2!} + \frac{1^2+2^2+3^2}{3!} \ldots\ldots\ldots + \frac{1^2+2^2+\cdots+n^2}{n!} + \ldots\ldots$

Solution.

Let the n^{th} term of the series be u_n and the sum to infinity be S.

Then $u_n = \frac{1^2+2^2+\cdots+n^2}{n!} = \frac{n(n+1)(2n+1)}{6} \frac{1}{n!}$

Let $n(n+1)(2n+1)$ = A + Bn + Cn (n – 1) + Dn (n – 1)(n – 2).

∴ A = 0, B = 6, C = 9, D = –2.

$$\therefore \; S = \sum_{n=1}^\infty \frac{6n+9n(n-1)+2n(n-1)(n-2)}{6} \frac{1}{n!}$$

$$= \sum_{n=1}^\infty \frac{1}{(n-1)!} + \frac{3}{2}\sum_{n=1}^\infty \frac{1}{(n-2)!} + \frac{1}{3}\sum_{n=1}^\infty \frac{1}{(n-3)!}$$

$= e + \frac{3}{2}e + \frac{1}{3}e$

$= \frac{17}{6}e.$

Exercises

1. Show that the sum to infinity of the series

   $2^2 + \frac{3^2}{1!}x + \frac{4^2}{2!}x^2 + \frac{5^2}{3!}x^3 + \ldots = e^x(x^2 + 5x + 4).$

2. Find the sum to infinity of the series

   (1) $\frac{3.5}{1!}x + \frac{4.6}{2!}x^2 + \frac{5.7}{3!}x^3 + \ldots\infty$

   (2) $1.2 + 2.3x + 3.4.\frac{x^2}{2!} + 4.5.\frac{x^3}{3!}\ldots$

3. Sum to infinity the following series:-

   (1) $1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \frac{1+2+3+4}{4!} + \ldots..$

   (2) $\frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + \ldots$

   (3) $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \ldots..$

   (4) $\frac{1.2}{1!} + \frac{2.3}{2!} + \frac{3.4}{3!} + \frac{4.5}{4!} + \ldots..$

4. Show that

   (1) $5 + \frac{2.6}{1!} + \frac{3.7}{2!} + \frac{4.8}{3!} + \ldots..$ to $\infty = 13e.$

   (2) $\frac{1^2.\ 2^2}{1!} + \frac{2^2.\ 3^2}{2!} + \frac{3^2.\ 4^2}{3!} + \ldots$ to $\infty = 27e.$

   (3) $\sum_{n=1}^{\infty} \frac{n^3 - n + 1}{n!} = 5e - 1.$

   Answers : 2.(1).$(x^2 + 7x + 8)\ e^x$, (2). $(x^2 + 4x + 2)e^x$, 3.(1).$\frac{3e}{2}$,(2).15e, (3).e + 1,(4). 3e.

Example 1. Sum the series $\sum_{n=1}^{\infty} \frac{n^2+3}{n+2} \cdot \frac{x^n}{n!}$.

Solution.

Let the sum of the series be S.

Then $S = \sum_{n=1}^{\infty} \frac{(n^2+3)(n+1)}{(n+2)!} \cdot x^n$.

Let $(n^2+3)(n+1) = \mathrm{A} + \mathrm{B}(n+2) + \mathrm{C}(n+2)(n+1) + \mathrm{D}(n+2)(n+1)n$.

We can easily find that A = −7, B = 7, C = −2 and D = 1.

Then $\mathrm{S} = \sum_{n=1}^{\infty} \frac{-7+7(n+2)-2(n+2)(n+1)+(n+2)(n+1)n}{(n+2)!} \cdot x^n$.

$$= -7\sum_{n=1}^{\infty}\frac{x^n}{(n+2)!} + 7.\sum_{n=1}^{\infty}\frac{x^n}{(n+1)!} - 2\sum_{n=1}^{\infty}\frac{x^n}{n!} + \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}$$

Now $\sum_{n=1}^{\infty}\frac{x^n}{(n+2)!} = \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{(n+2)!} + \ldots$

$$= \frac{1}{x^2}\left(e^x - 1 - x - \frac{x^2}{2!}\right).$$

$$\sum_{n=1}^{\infty}\frac{x^n}{(n+1)!} = \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots + \frac{x^n}{(n+1)!} + \ldots$$

$$= \frac{1}{x}(e^x - 1 - x).$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n!} = \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = e^x - 1.$$

$$\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = x + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{(n-1)!} + \ldots = xe^x$$

$$\therefore\ \mathrm{S} = \frac{-7}{x^2}\left(e^x - 1 - x - \frac{x^2}{2!}\right) + \frac{7}{x}(e^x - 1 - x) - 2\,(e^x - 1) + xe^x$$

$$= \frac{e^x}{x^2}(x^3 - 2x^2 + 7x - 7) + \frac{7}{2x^2}(3x^2 + 2).$$

Example 2. Sum the series $\frac{5}{1!} + \frac{7}{3!} + \frac{9}{5!} + \ldots$

Solution.

The n^{th} term $\mathrm{u}_n = \frac{(2n+3)}{(2n-1)!}$

Put $2n + 3 = \mathrm{A}(2n-1) + \mathrm{B}$.

Then A = 1 and B = 4.

$$\therefore \ u_n = \frac{2n-1+4}{(2n-1)!}$$

$$= \frac{2n-1}{(2n-1)!} + \frac{4}{(2n-1)!}$$

$$= \frac{1}{(2n-2)!} + \frac{4}{(2n-1)!}$$

$$\therefore \ u_1 = 1 + \frac{4}{1!}$$

$$u_2 = \frac{1}{2!} + \frac{4}{3!}$$

$$u_3 = \frac{1}{4!} + \frac{4}{5!}$$

…………..

…………..

Sum to infinity = $\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots \right) + 4\left(\frac{1}{1!} + \frac{1}{3!} + \ldots \right)$

$$= \frac{1}{2}\left(e + \frac{1}{e}\right) + 4 \cdot \frac{1}{2} \cdot \left(e - \frac{1}{e}\right)$$

$$= \frac{5}{2}e - \frac{3}{2e}.$$

Example 3. Prove that the infinite series $\frac{2\frac{1}{2}}{1!} - \frac{3\frac{1}{3}}{2!} + \frac{4\frac{1}{4}}{3!} - \frac{5\frac{1}{5}}{4!} + \ldots.. = \frac{1+e}{e}.$

Solution.

Let u_n be the n^{th} term of the series and S be the sum of the series to infinity.

Then $u_n = (-1)^{n+1} \frac{(n+1)\frac{1}{n+1}}{n!}$

$$= (-1)^{n+1} \frac{(n+1)^2+1}{(n+1)!}.$$

Put $n^2 + 2n + 2$ = A + B (n+1) + C (n+1) n.

$\therefore$ A = 1 , B = 1 , C = 1.

$\therefore$ $u_n = (-1)^{n+1} \frac{1+(n+1)+(n+1)n}{(n+1)!}$

$= (-1)^{n+1} . \left\{\frac{1}{(n+1)!} + \frac{1}{n!} + \frac{1}{(n-1)!}\right\}.$

$\therefore$ S = $\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n+1)!} + \sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{n!} + \sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n-1)!}$

Now $\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n+1)!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots. = e^{-1}$

$\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} \ldots = -e^{-1} + 1$

$\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{1}{(n-1)!} = 1 - \frac{1}{1!} + \frac{1}{2!} \ldots = e^{-1}.$

$\therefore$ S = $1 + e^{-1}$

$= \frac{e+1}{e}$.

Exercises

1. Show that

   (1) $\sum_{n=1}^{\infty} \frac{n-1}{n+2} . \frac{x^n}{n!} = \frac{1}{x^2}\{(x^2 - 3x - 3)\ e^x + \frac{1}{2}x^2 - 3\}.$

   (2) $\sum_{n=1}^{\infty} \frac{(2n-1)}{(n+3)n!} = \frac{1}{2}(43 - 15e)$

2. Sum to infinity the series

   (1) $\frac{3}{1!} + \frac{4}{3!} + \frac{5}{5!} + \frac{6}{7!} + \ldots..$

   (2) $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \ldots..$

   (3) $\frac{3}{2!} + \frac{5}{4!} + \frac{7}{6!} + \frac{9}{8!} + \ldots..$

3. Show that $\sum_{0}^{\infty} \frac{5n+1}{(2n+1)!} = \frac{e}{2} + \frac{2}{e}$

4. Prove that $\frac{2^2}{1!} + \frac{2^4}{3!} + \frac{2^6}{5!} = \frac{e^4-1}{e^2}.$

5. Show that $\log_e 2 - \frac{1}{2!}(\log_e 2)^2 + \frac{1}{3!}(\log_e 2)^3 \ldots\ldots = \frac{1}{2}$.

Answer : 2(1).$\frac{1}{e}$,(2).$\frac{1}{2}(3e - 2e^{-1})$,(3).$\frac{1}{2e}$.

By equating the coefficients of like powers of x in the expansions of function of x in two different ways, we can derive some identities. The following examples will illustrated the method:

Example 1. By expanding $(e^x - 1)^n$ in two ways or otherwise prove that

$n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r - \ldots\ldots = 0$ where r < n.

What is the sum of the above series when r = n?

Solution.

$$(e^x - 1)^n = e^{nx} - {}_nC_1 e^{(n-1)x} + \ldots.$$

$$= 1 + nx + \frac{(nx)^2}{1!} + \ldots \frac{(nr)^r}{r!} + \ldots - {}_nC_1\left[1 + (n-1)x + \frac{\{(n-1)x\}^2}{2!} + \cdots \frac{\{(n-1)x\}^r}{r!} + \right.$$

…

$$\left. + {}_nC_2\left[1 + (n-2)x + \frac{\{(n-2)x\}^2}{2!} + \cdots \frac{\{(n-2)x\}^r}{r!} + \cdots\right] \ldots \right.$$

Coefficient of x^r in the expansion of $(e^x - 1)^n$

$$= \frac{n^r}{r!} - {}_nC_1.\frac{(n-1)^r}{r!} + {}_nC_2.\frac{(n-2)^r}{r!} - \ldots\ldots$$

$$= \frac{1}{r!}\{n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r \ldots.\}$$

Again $(e^x - 1)^n = (1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots - 1)^n$

$$= \left(\frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \cdots\right)^n$$

$$= x^n\left(\frac{1}{1!} + \frac{x}{2!} + \cdots \frac{x^{n-1}}{n!} + \cdots\right)^n.$$

All terms in the expansion contain x^n and the higher power of x.

$\therefore$ If $r < n$, there will be no term containing x^r in the expansion.

$\therefore \ \frac{1}{r!}\{n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r \ldots.\} = 0$

i.e., $n^r - {}_nC_1(n-1)^r + {}_nC_2(n-2)^r \ldots. = 0$

If $r = n$, then

$\frac{1}{n!}\{n^n - {}_nC_1(n-1)^n + {}_nC_2(n-2)^n \ldots.\}$

$= \text{Coefficient of } x^n \text{ in the expansion of } x^n\left(\frac{1}{1!} + \frac{x}{2!} + \cdots\right)^n$

$= 1$.

$\therefore \ n^n - {}_nC_1(n-1)^n + {}_nC_2(n-2)^n \ldots. = n!$

Example 2. Show that if a^r be the coefficient of x^n in the expansion of $e^{e\,x}$, then

$a_r = \frac{1}{r!}\left\{\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!}\right\}.$

Hence show that

$(i)\ \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \ldots = 5e$

$(ii)\ \frac{1^4}{1!} + \frac{2^4}{2!} + \frac{3^4}{3!} + \ldots = 15e.$

Solution.

$e^{e\,x} = 1 + e^x + \frac{(e^x)^2}{2!} + \frac{(e^x)^3}{3!} + \frac{(e^x)^4}{4!} + \ldots.$

$= 1 + e^x + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + \frac{e^{4x}}{4!} + \ldots.$

$= 1 + (1 + x + \frac{x^2}{2!} + \cdots \frac{x^r}{r!} + \ldots.) + \frac{1}{2!}(1 + 2x + \frac{2^2x^2}{2!} + \cdots \frac{2^rx^r}{r!} + \ldots.)$

$+ \frac{1}{3!}(1 + 3x + \frac{3^2x^2}{2!} + \cdots \frac{3^rx^r}{r!} + \ldots.) + \ldots$

Hence the coefficient of $x^r = \frac{1}{r!}\left\{\frac{1^r}{1!} + \frac{2^r}{2!} + \frac{3^r}{3!}\right\}$.

Again

$$e^{e^x} = e^{1+x+\frac{x^2}{2!}+\cdots} = e.\, e^{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$$= e.\{1 + (x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) + \frac{1}{2!}(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^2$$

$$+ \frac{1}{3!}(x + \frac{x^2}{2!} + \cdots)^3 + \ldots\}$$

Coefficient of $x^3 = e\left(\frac{1}{3!} + \frac{1}{2!}.2.\frac{1}{2!} + \frac{1}{3!}\right)$

$$= \frac{e}{3!}(1 + 3 + 1) = \frac{5e}{3!}.$$

We have shown that the coefficient of x^3

$$= \frac{1}{3!}\left(\frac{1^3}{1!} + \frac{2^3}{2!} + \cdots\right)$$

$$\therefore \quad \frac{1}{3!}\left(\frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \cdots\right) = \frac{5e}{3!}$$

$$\therefore \quad \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \ldots.. = \frac{5e}{3!}.$$

Similarly equating the coefficient of x^4 , we get the second result.

Example 3. Prove that if n is a positive integer

$$1 - \frac{n}{1^2}x + \frac{n(n-1)}{1^2.2^2}x^2 - \frac{(n-1)(n-2)}{1^2.2^2.3^2}x^3 + \cdots$$

$$= e^x\{1 - \frac{n+1}{1^2}x + \frac{(n+1)(n+2)}{1^2.2^2}x^2 - \frac{(n+1)(n+2)(n+3)}{1^2.2^2.3^2}x^3 + \ldots.\}.$$

Solution.

$$e^y = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots..$$

$$(1-\frac{x}{y})^n = 1 - n.\frac{x}{y} + \frac{n(n-1)}{2!}.(\frac{x}{y})^2 - \frac{n(n-1)(n-2)}{3!}(\frac{x}{y})^3 + \ldots..$$

$$\therefore\ 1 - \frac{n}{1^2}x + \frac{n(n-1)}{1^2.2^2}x^2 - \frac{(n-1)(n-2)}{1^2.2^2.3^2}x^3 + \cdots$$

= the term independent of y in the product of $e^y(1-\frac{x}{y})^n$.

$$e^y(1-\frac{x}{y})^n = e^x.\, e^{y-x}.\, \frac{(y-x)^n}{y^n}$$

$$= e^x.\{\ 1 + \frac{(y-x)}{1!} + \frac{(y-x)^2}{2!} + \ldots.\}\ \frac{(y-x)^n}{y^n}$$

$$= e^x \left\{\frac{(y-x)^n + \frac{(y-x)^{n+1}}{1!} + \frac{(y-x)^{n+2}}{2!} + \cdots}{y^n}\right\}$$

The term containing y^n in the expression

$$(y-x)^n + \frac{(y-x)^{n+1}}{1!} + \frac{(y-x)^{n+2}}{2!} + \cdots$$

is $y^n - \frac{{}_{n+1}C_1}{1!}y^n.x + \frac{{}_{n+2}C_2}{2!}y^n\ x^2 \ldots.$

$\therefore$ Term independent of y in $e^y(1-\frac{x}{y})^n$ is

$$e^x\{1 - \frac{{}_{n+1}C_1}{1!}x + \frac{{}_{n+2}C_2.x^2}{2!} - \ldots.\}$$

$$= e^x\{1 - \frac{(n+1)}{(1!)^2}x + \frac{(n+2)(n+1)}{(2!)^2}x^2 - \ldots.\}.$$

Hence the required result.

Exercises

1. Show that , if n is a positive integer

   $n\,.\,1^{n+1} - \frac{n(n-1)}{2!}.2^{n+1} + \frac{n(n-1)(n-2)}{3!}.3^{n+1} - \ldots. = (-1)^n.n.\frac{(n+1)!}{2}.$

2. Find the coefficient of x^r in the expansion of $\frac{e^{nx}-1}{1-e^{-x}}$, n being a positive integer and find the values of

(1) $1^2 + 2^2 + 3^2 + \ldots.. + n^2$

(2) $1^3 + 2^3 + 3^3 + \ldots.. + n^3$

(3) $1^4 + 2^4 + \ldots.. + n^4$

3. By means of the identity $e^{x^2+\frac{1}{x^2}+2} = e^{(x+\frac{1}{x})^2}$ show that

$$e^2\{1 + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \frac{1}{(3!)^2} \ldots.\} = 1 + \frac{2!}{(1!)^3} + \frac{4!}{(2!)^3} + \frac{6!}{(3!)^3} + \ldots.$$

[Left side = term independent of x in $e^2.e^{x^2}.e^{x^{\frac{1}{2}}}$

$$e^{(x+\frac{1}{x})^2} = 1 + \frac{(x+\frac{1}{x})^2}{1!} + \frac{(x+\frac{1}{x})^4}{2!} + \frac{(x+\frac{1}{x})^6}{4!} + \ldots\ldots$$

Term independent of x in the above expansion

$$= 1 + \frac{{}_2C_1}{1!} + \frac{{}_4C_2}{2!} + \frac{{}_6C_3}{3!} + \ldots..]$$

Answer : 2(1).$\frac{n(n+1)(2n+1)}{6}$,(2).$\frac{n^2(n+1)^2}{4}$,(3).$\frac{n(n+1)(6n^3+9n^2+n-1)}{60}$.

Logarithmic series

$$\log(1+x) = x - \frac{x^2}{2!} + \frac{1.2}{3!}x^3 - \frac{1.2.3}{4!}x^4 \ldots..$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots..$$

Modification of the logarithmic series.

If $-1 < x < 1$, we have

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots.. + (-1)^{n-1}\frac{x^n}{n} + \ldots \qquad \ldots\ldots\ldots(1)$$

It is convenient to remember the form of the series in the case in which x is negative.

Thus

$\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \ldots\ldots$

$= -(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots)$

i.e., $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$ ……….(2)

Adding the series (1) and (2),

$\log(1+x) - \log(1-x) = 2x + 2.\frac{1}{3}x^3 + 2.\frac{1}{5}x^5 + \ldots\ldots$

i.e., $\log\frac{1+x}{1-x} = 2\,(x + \frac{x^3}{2} + \frac{x^5}{3} + \cdots)$

$\log(1+x) + \log(1-x) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots\right)$

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Using the different forms of the logarithmic series we can find the sums of the certain series. The following examples will illustrate the methods of such summation.

Example 1. Show that if $x > 0$. $\log x = \frac{x-1}{x+1} + \frac{1}{2}.\frac{x^2-1}{(x+1)^2} + \frac{1}{3}.\frac{x^3-1}{(x+1)^3} + \ldots..$

Solution.

R.H.S. $= \frac{x}{x+1} + \frac{1}{2}.\left(\frac{x}{x+1}\right)^2 + \frac{1}{3}.\left(\frac{x}{x+1}\right)^3 \ldots.. - \left\{\frac{1}{x+1} + \frac{1}{2}.\frac{1}{(x+1)^2} + \frac{1}{3}.\frac{1}{(x+1)^3} \ldots\right\}$

$= -\log\,(1 - \frac{x}{x+1}) + \log\,(1 - \frac{x}{x+1})$

$= -\log\frac{1}{x+1} + \log\frac{x}{x+1}$

$= \log\left\{\left(\frac{x}{x+1}\right) + \frac{1}{x+1}\right\}$

$= \log x$.

The expansion is valid when

$\left|\frac{x}{x+1}\right| < 1$ and $\left|\frac{1}{x+1}\right| < 1$, $\left|\frac{x}{x+1}\right|$ is always less than 1.

When $\left|\frac{1}{x+1}\right| < 1$, $|x+1| > 1$, i.e., $|x| > 0$

∴ When $x > 0$, the expansion is valid .

Example 2. Show that $\log\sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \ldots.$

Solution.

Right side expression can be written as

$$\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \ldots + 1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} + \frac{1}{7}\cdot\frac{1}{4^3} + \ldots..$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{6}\cdot\left(\frac{1}{2}\right)^6 + \ldots.. + 1 + \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{5}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{7}\cdot\left(\frac{1}{2}\right)^6 + \ldots.$$

$$= \frac{1}{2}\cdot x^2 + \frac{1}{4}x^4 + \frac{1}{6}x^6 + \ldots.. + 1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 + \ldots. \text{ When } x = \frac{1}{2}$$

$$= \frac{1}{2}\left\{x^2 + \frac{1}{2}\cdot x^4 + \frac{1}{3}x^6 + \ldots..\right\} + \frac{1}{x}\left\{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \ldots\right\}$$

$$= -\frac{1}{2}\log(1-x^2) + \frac{1}{2x}\log\frac{1+x}{1-x}.$$

∴ The series $= -\frac{1}{2}\log\left(1-\frac{1}{4}\right) + \log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}$, since $x = \frac{1}{2}$.

$$= -\frac{1}{2}\log\frac{3}{4} + \log 3$$

$$= \frac{1}{2}\log 9 - \frac{1}{2}\log\frac{3}{4}$$

$$= \frac{1}{2}\log\left(\frac{9.4}{3}\right)$$

$$= \frac{1}{2}\log 12$$

$$= \log\sqrt{12}.$$

Example 3. If a , b , c denote three consecutive integers, show that

$$\log_e b = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2ac+1} + \frac{1}{3}\cdot\frac{1}{(2ac+1)^3} + \ldots.$$

Solution.

$$\text{Right side} = \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{1+\frac{1}{2ac+1}}{1-\frac{1}{2ac+1}}$$

$$= \frac{1}{2}\log_e a + \frac{1}{2}\log_e c + \frac{1}{2}\log_e \frac{2ac+1}{2ac}$$

$$= \frac{1}{2}\log(ac) + \frac{1}{2}\log\frac{ac+1}{ac}$$

$$= \frac{1}{2}\log ac\,.\,\frac{ac+1}{ac}$$

$$= \frac{1}{2}\log(ac+1).$$

If a , b , c denote three consecutive integers then $b = a + 1$ and $b = c - 1$

$\therefore\ a = b - 1$; $c = b + 1$.

$\therefore\ ac = b^2 - 1$, i.e., $ac + 1 = b^2$.

$\therefore\ \frac{1}{2}\log(ac+1) = \frac{1}{2}\log(b^2) = \log b.$

Exercises

1. Show that

$$\log\frac{a+x}{a-x} = \frac{2ax}{a^2+x^2} + \frac{1}{3}\cdot\left(\frac{2ax}{a^2+x^2}\right)^3 + \frac{1}{5}\cdot\left(\frac{2ax}{a^2+x^2}\right)^5 + \ldots..$$

2. Sum the series $\frac{1}{2x-1} + \frac{1}{3}\cdot\frac{1}{(2x-1)^3} + \frac{1}{5}\cdot\frac{1}{5(2x-1)^5} + \ldots.$

3. Show that when $-1 < x < \frac{1}{3}$

$$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) = \frac{2x}{1-x} - \frac{1}{2}\cdot\left(\frac{2x}{1-x}\right)^2 + \frac{1}{3}\cdot\left(\frac{2x}{1-x}\right)^3 \ldots.$$

4. Show that

$$\log(x+2h) = 2\log(x+h) - \log(x) - \left\{\frac{h^2}{(x+h)^3} + \frac{h^4}{2(x+h)^3} + \frac{h^6}{3(x+h)^3} + \cdots\right\}.$$

5. Show that

$$\log_e\left(1+\frac{1}{n}\right)^2 = 1 - \frac{1}{2(n+1)} - \frac{1}{2.3(n+1)^2} - \frac{1}{3.4(n+1)^3} \ldots.. \infty.$$

6. Show that $\log_e 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots..$

7. Sum the series $(1+\frac{1}{2}) + \left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9} + \left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2} + \ldots..$ to infinity.

8. Sum to infinity the series $\sum\left(\frac{1}{2n+1} + \frac{1}{(2n)!}\right)x^{2n+1}$, $(x^2 < 1)$.

9. Prove that $\sum_1^\infty \frac{1}{2n-1}\left(\frac{1}{9^{n-1}} + \frac{1}{9^{2n-1}}\right) = \frac{1}{2}\log_e 10.$

Answer : 2. $\frac{1}{2}log\left(\frac{x}{x-1}\right)$, 7. 9log 3 – 12log 2, 8. $\frac{1}{2}\left[log\frac{1+x}{1-x} + x(e^x + e^{-x})\right]$.

Series which can be summed up by the logarithmic series.

We can split the general term into partial fractions and using the result

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$ We can sum certain series. The following examples will illustrate the method.

Example 1. Sum the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)2n(2n+1)}$.

Solution.

Let S be the sum of the series and u_n be the n^{th} term.

Then $u_n = \frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2}\cdot\frac{1}{2n+1}$

$\therefore\ u_1 = \frac{1}{2}\cdot\frac{1}{1} - \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}$

$u_2 = \frac{1}{2}\cdot\frac{1}{3} - \frac{1}{4} + \frac{1}{2}\cdot\frac{1}{5}$

$u_3 = \frac{1}{2}\cdot\frac{1}{5} - \frac{1}{6} + \frac{1}{2}\cdot\frac{1}{7}$

……………………………………….

……………………………………..

Adding the last fraction of a term with the first fraction of the next term, we get

$$S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \ldots..$$

$$= -\frac{1}{2} + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots.$$

$$= -\frac{1}{2} + \log 2.$$

Example 2. Show that $\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \ldots. \infty = 3 \log 2 - 1.$

Solution.

Let S be the sum of the series and u_n be the n^{th} term of the series.

Then $u_n = \frac{2n+3}{(2n-1)(2n+1)}$.

Splitting u_n into partial fractions, we get

$$u_n = 2 \,.\, \frac{1}{2n-1} - 3 \,.\, \frac{1}{2n} + 1 \,.\, \frac{1}{2n+1}$$

Giving values 1 , 2 , 3 , …. in u_n , we have

$$u_1 = 2 \,.\frac{1}{1} - 3 \,.\frac{1}{2} + 1 \,.\frac{1}{3}$$

$$u_2 = 2 \,.\frac{1}{3} - 3 \,.\frac{1}{4} + 1 \,.\frac{1}{5}$$

$$u_3 = 2 \,.\frac{1}{5} - 3 \,.\frac{1}{6} + 1 \,.\frac{1}{7}$$

……………………………………….

……………………………………..

$$\therefore \; S = 2 - 3 \,.\frac{1}{2} + 3 \,.\frac{1}{3} - 3 \,.\frac{1}{4} + 3 \,.\frac{1}{5} \ldots.$$

$$= 2 + 3\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right)$$

$= 2 + 3(1 - \frac{1}{2} + \frac{1}{3} \ldots\ldots - 1)$

$= 2 + 3 (\log 2 - 1)$

$= -1 + 3 \log 2.$

Exercises

Show that the sum of the series to infinity

1. $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{3.6} + \ldots. = \log 2$
2. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots. = 2 - \log 2.$
3. $\frac{1}{1.2.3} + \frac{5}{3.4.5} + \frac{9}{5.6.7} + \frac{13}{7.8.9} + \ldots. = \frac{5}{2} - 3 \log 2.$
4. $\frac{1}{2.3.4} + \frac{5}{4.5.6} + \frac{9}{6.7.8} + \ldots. = \frac{3}{4} - \log 2$

If k is a positive integer and $|x| < 1$, then

$$\sum_{n=1}^{\infty} \frac{x^2}{n+k} = \frac{x}{1+k} + \frac{x^2}{2+k} + \frac{x^3}{3+k} + \frac{x^4}{4+k} + \ldots\ldots$$

$$= \frac{1}{x^k}\left(\frac{x^{k+1}}{k+1} + \frac{x^{k+2}}{k+2} + \frac{x^{k+3}}{k+3} + \cdots \infty\right)$$

$$= \frac{1}{x^k} \left\{ x + \frac{x^2}{2} + \ldots.+ \frac{x^k}{k} + \frac{x^{k+1}}{k+1} + \frac{x^{k+2}}{k+2} + \frac{x^{k+3}}{k+3} + \ldots \infty \right.$$

$$\left. -(x + \frac{x^2}{2} + \ldots.+ \frac{x^k}{k}) \right\}$$

$$= \frac{1}{x^k} \{ - \log (1 - x) -(x + \frac{x^2}{2} + \ldots.+ \frac{x^k}{k}) \}$$

$$= -\frac{1}{x^k} \{ \log (1 - x) + x + \frac{x^2}{2} + \ldots.+ \frac{x^k}{k} \}$$

Similarly $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = -\frac{1}{x} \{ \log (1 - x) + x \}$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+2} = -\frac{1}{x^2} \{ \log (1 - x) + x + \frac{x^2}{2} \}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+3} = -\frac{1}{x^3} \{ \log (1 - x) + x + \frac{x^2}{2} + \frac{x^3}{3} \}$$

Using these result we can sum certain series. The following examples will illustrate the method.

Example 1. Sum the series $\sum_{n=1}^{\infty} \frac{n^3+n^2+1}{n(n+2)} x^n$ when $|x| < 1$.

Solution.

Split $\frac{n^3+n^2+1}{n(n+2)}$ into partial fractions.

We have $S = \sum_{n=1}^{\infty}\{ (n-1) + \frac{1}{2} . \frac{1}{n} + \frac{3}{2} . \frac{1}{n+2} \} x^n$

$= \sum_{n=1}^{\infty}\{ (n-1)x^n + \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^n}{n} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{x^n}{n+2}$.

$\sum_{n=1}^{\infty}\{ (n-1)x^n = x^2 + 2x^3 + 3x^4 + \ldots.. \infty$

$= x^2 (1 + 2x + 3x^2 + \ldots. \infty)$

$= x^2 (1-x)^2 = \frac{x^2}{(1-x)^2}$.

$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$.

$\sum_{n=1}^{\infty} \frac{x^n}{n+2} = -\frac{1}{x^2} \{ \log(1-x) + x + \frac{x^2}{2} \}$

$\therefore$ $S = \frac{x^2}{(1-x)^2} - \frac{1}{2}\log(1-x) - \frac{3}{2x^2} \{ \log(1-x) + x + \frac{x^2}{2} \}$.

Example 2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)(n+2)}$.

Solution.

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2} . \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2} . \frac{1}{n+2}$$

Let S be the sum of the series

$S = \sum_{1}^{\infty} \left(\frac{1}{2} . \frac{1}{n} - \frac{1}{n+1} + \frac{1}{2} . \frac{1}{n+2}\right) (-1)^{n+1} x^n$

$$= \frac{1}{2} \cdot \sum_1^\infty \frac{(-1)^{n+1} x^n}{n} - \sum_1^\infty \frac{(-1)^{n+1} x^n}{n+1} + \frac{1}{2} \sum_1^\infty \frac{(-1)^{n+1} x^n}{n+2}.$$

We have

$$\sum_1^\infty \frac{(-1)^{n+1} x^n}{n} = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} \dots\dots = \log(1+x)$$

$$\sum_1^\infty \frac{(-1)^{n+1} x^n}{n+1} = \frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} \dots\dots = \frac{1}{x}\left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} \dots.\right)$$

$$= \frac{1}{x}\{-\log(1+x) + x\}$$

$$\sum_1^\infty \frac{(-1)^{n+1} x^n}{n+2} = \frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} \dots\dots = \frac{1}{x^2}\left\{\frac{x}{3} - \frac{x^2}{4} + \frac{x^3}{5} \dots.\right\}$$

$$= \frac{1}{x^2}\left\{\log(1+x) - x + \frac{x^2}{2}\right\}.$$

$$\therefore\ S = \frac{1}{2}\log(1+x) - \frac{1}{x}\{-\log(1+x) + x\} + \frac{1}{2x^2}\left\{\log(1+x) - x + \frac{x^2}{2}\right\}$$

$$= \frac{1}{2}\log(1+x)\left(1 + \frac{2}{x} + \frac{1}{x^2}\right) - \left(\frac{3}{4} + \frac{1}{2x}\right).$$

Exercises

1. Prove that the sum of the infinite series whose n^{th} term is $\frac{1}{n(n+1)} \cdot \frac{1}{2^n}$ is $1 - \log 2$.
2. Sum the series

   (1) $\sum_1^\infty \frac{n^2+1}{n(n+2)} x^n$.

   (2) $\sum_1^\infty \frac{(n+1)^3}{n(n+3)} x^n$.

   (3) $\sum_1^\infty \frac{n^2}{(n+1)(n+2)} x^n$.
3. Show that

   (1) $\frac{3}{1.2.2} - \frac{4}{2.3.2^2} + \frac{5}{3.4.2^3} - \dots. = 4\log\frac{3}{2} - 1$.
4. Show that

   (1) $\sum_{r=1}^\infty \frac{4r-1}{2r(2r-1)} \cdot \frac{1}{3^{2r}} = \log 3 - \frac{4}{3}\log 2$.

(2) $\sum_{r=1}^{\infty} \frac{10r+1}{2r(2r-1)(2r+1)} \cdot \frac{1}{2^{2r}} = 2 - \log 2 - \frac{3}{4} \log 3.$

Answer : 2(1).$\frac{5-x^2}{2x^2}\log(1-x) + \frac{(x^2+5x-10)}{4x(x-1)}$,(2).$\frac{x}{(1-x)^2} - \frac{4}{9}(2x^3+3x^2+6x) - 3\log(1-x)$,(3).$\frac{9-4x^3}{12x^2}\log(1-x) + \frac{6x^3-x^2-3x+6}{8x(1-x)}$.

Calculation of logarithms by means of the logarithmic series.

The direct calculation of logarithms by means of the series

$$\text{Log}\,(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\ldots \infty$$

is somewhat tedious , since the series is slowly convergent, i.e., very many terms of the series have to be calculated before a given degree of approximation is attained.

The calculation is usually carried out in practice as follows.

We have proved that

$$\log_e \frac{1+x}{1-x} = 2\left\{x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right\}$$

When $-1 < x < 1$.

Let $y = \frac{1+x}{1-x}$, i.e., $x = \frac{y-1}{y+1}$.

$$\therefore \log_e y = 2\left\{\frac{y-1}{y+1} + \frac{1}{3}\cdot\left(\frac{y-1}{y+1}\right)^3 + \frac{1}{5}\cdot\left(\frac{y-1}{y+1}\right)^5 + \cdots\right\}$$

Where y lies between 0 and $+\infty$.

Put $y = \frac{p}{q}$ in this series where p and q are positive integers.

$$\therefore \log_e p - \log_e q = 2\left\{\left(\frac{p-q}{p+q}\right) + \frac{1}{3}\cdot\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\cdot\left(\frac{p-q}{p+q}\right)^5 + \ldots\ldots\right.$$

Now if p and q be fairly large and differ little in value, i.e., $(p - q)$ is small, the above series converges rapidly to the limits, since the terms become small quickly.

Example. Evaluate log 2 to 5 places of decimals.

Solution.

Put $p = 2$, $q = 1$.

$\therefore \log_e 2 - \log_e 1 = 2 . \left(\frac{1}{3} + \frac{1}{3} . \frac{1}{3^3} + \frac{1}{5} . \frac{1}{5^3} + \cdots\right)$

$\log_e 1 = 0$.

$\frac{1}{3} = 0.333,333,3$ $\frac{1}{3^3} = 0.037,037$ $\frac{1}{3} . \frac{1}{3^3} = 0.012,345,7$ $\frac{1}{3^5} = 0.004,115,2$

$\frac{1}{5} . \frac{1}{3^5} = 0.000,832,0$ $\frac{1}{3^7} = 0.000,457,2$ $\frac{1}{7} . \frac{1}{3^7} = 0.000,055,3$ $\frac{1}{3^9} = 0.000,050,8$

$\frac{1}{9} . \frac{1}{3^9} = 0.000,005,6$ $\frac{1}{3^{11}} = 0.000,005,6$ $\frac{1}{11} . \frac{1}{3^{11}} = 0.000,000,5$

$\therefore$ Sum of the first 6 terms is 2 (0.346,573,4) approximately

i.e., 0.693,146,8

$\therefore$ log 2 = 0.69315 to 5 places of decimals.

We can calculate the error involved in taking only the first six terms.

The difference between log 2 and the sum of the first six terms.

$$= 2\left\{\frac{1}{13} . \frac{1}{3^{13}} + \frac{1}{15} . \frac{1}{5^{15}} + \cdots\right\}$$

$$< \frac{2}{13}\left\{\frac{1}{3^{13}} + \frac{1}{3^{15}} + \cdots \infty\right\}$$

$$< \frac{2}{13} . \frac{1}{3^{13}} . \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots \infty\right)$$

$$< \frac{2}{13} . \frac{1}{3^{13}} . \frac{1}{1 - \frac{1}{3^2}}$$

$$< \frac{2}{13} . \frac{1}{3^{13}} . \frac{9}{8}$$

$$< \frac{1}{13} . \frac{1}{3^{11}} . \frac{1}{4}$$

$< \frac{1}{52}$. (0.0000056)

$< 0.0000011.$

Hence if we take log 2 = 0.69315, there is no error until the 6th place of decimals.

By means of this series by putting $p = 5$, $q = 4$, $\log_e 3$ can be calculated.

By putting $p = 5$, $q = 4$, $\log_e 3$ can be calculated.

Similarly we can calculated logarithms of numbers.

The application of the exponential and logarithmic series to limits and approximations.

The application is shown in the following examples:

Example 1. Evaluate $Lt_{x \to 0} \frac{e^x - e^{-x}}{\log(1+x)}$.

Solution.

$$Lt_{x \to 0} \frac{e^x - e^{-x}}{\log(1+x)} .$$

$$= Lt_{x \to 0} \frac{(1 + x + \frac{x^2}{2!} ....) - (1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} ....)}{x - \frac{x^2}{2!} + \frac{x^3}{3!} ...}$$

$$= Lt_{x \to 0} \frac{2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots}{x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

$$= Lt_{x \to 0} \frac{2 + \frac{2x}{3!} + \frac{2x^4}{5!} + \cdots}{1 - \frac{x}{2!} + \frac{x^2}{3!} - \cdots}$$

$= 2.$

Example 2. Evaluate $Lt_{n \to \infty} \left(1 + \frac{3}{n^2} + \frac{5}{n^3}\right)^{n^2 + 7n}$

Solution.

Let the value of the limit be A.

$\therefore$ A = $Lt_{n\to\infty}$ $(1 + \frac{3}{n^2} + \frac{5}{n^3})^{n^2+7n}$

Taking logarithms on both sides, we have

log A = $Lt_{n\to\infty}$ $(1 + \frac{3}{n^2} + \frac{5}{n^3})^{n^2+7n}$

$= Lt_{n\to\infty} \,(n^2 + 7n)(1 + \frac{3}{n^2} + \frac{5}{n^3})$

$= Lt_{n\to\infty} (n^2 + 7n) \{ \left(\frac{3}{n^2} + \frac{5}{n^3}\right) - \frac{1}{2}\left(\frac{3}{n^2} + \frac{5}{n^3}\right)^2 + \frac{1}{3}\left(\frac{3}{n^2} + \frac{5}{n^3}\right)^3 - \cdots\}$

$= Lt_{n\to\infty} (n^2 + 7n) \{ \left(\frac{3}{n^2} + \frac{5}{n^3}\right) - \frac{1}{2n^4}\left(3 + \frac{5}{n}\right)^2 + \frac{1}{3n^6}\left(3 + \frac{5}{n}\right)^3 \ldots..\}$

$= \underset{n\to\infty}{Lt} \{ 3 + \frac{5}{n} + \frac{21}{n} + \frac{35}{n^2} - \frac{1}{2n^2}\left(3 + \frac{5}{n}\right)^2 - \frac{7}{2n}\left(3 + \frac{5}{n}\right)^2 + \ldots.\}$

Except the first , all the other term will contain $\frac{1}{n}$ or higher powers of $\frac{1}{n}$.

$\therefore$ log A = 3.

$\therefore$ A = e^3.

Example 3. Prove that, if n is large $\left(n - \frac{1}{3n}\right) \log \frac{n+1}{n-1} = 2 + \frac{8}{45n^4} + \ldots.$

and $\left(\frac{n+1}{n-1}\right)^{n-\frac{1}{3n}} = e^2 \left(1 + \frac{8}{45n^4} + \ldots\right)$

Solution.

Let $\left(\frac{n+1}{n-1}\right)^{n-\frac{1}{3n}}$ be A.

$\therefore$ log A = $\left(n - \frac{1}{3n}\right) \log \frac{n+1}{n-1}$

$= \left(n - \frac{1}{3n}\right) \log \frac{1+\frac{1}{n}}{1-\frac{1}{n}}$

$= \left(n - \frac{1}{3n}\right) \{ \log (1 + \frac{1}{n}) - \log (1 - \frac{1}{n}) \}$

$$= 2\left(n - \frac{1}{3n}\right)\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} + \ldots\right\}$$

$$= 2\left\{1 + \frac{1}{3n^2} + \frac{1}{5n^4} + \frac{1}{7n^6} + \ldots - \frac{1}{3n^2} - \frac{1}{9n^4}\ldots\right\}$$

$$= 2\left\{1 + \frac{4}{45n^4} + \ldots\right\}$$

$$= 2 + \frac{8}{45n^4}$$

$$\therefore \quad A = e^{2+\frac{8}{45n^4}}$$

$$= e^2.\left\{1 + \frac{8}{45n^4} + \ldots\right\}$$

Example 4. Show that if $e^x = 1 + xe^{yx}$, where x^3 and higher powers of x can be neglected,

$$y = \frac{1}{2!} + \frac{1}{4!}$$

Solution.

Now $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots..$

$$\therefore \quad e^x - 1 = x\left\{1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right\}$$

$$\therefore \quad xe^{yx} = x\left\{1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right\}$$

$$\therefore \quad e^{yx} = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Taking logarithms on both sides, we have

$$yx = \log\left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^4}{4!} + \ldots\right) - \frac{1}{2}\left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots\right)^2$$

Exercises

1. Evaluate $\lim_{x\to 0} \frac{e^x - \log(e+ex)}{x^2}$.

2. Evaluate $\lim_{x\to 0} \frac{e^x - \log_e(1+x)(1+2x)}{5x^3}$.

3. Find $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$.

4. Find the limit as $x \to 1$ of $\frac{\log x}{x^2-3x+2}$.

5. Evaluate $\lim_{x\to 0} \frac{(2+x)\log(1+x)+(2-x)\log(1-x)}{x^4}$.

6. Evaluate $\lim_{n\to\infty}(1+\frac{3}{n^2}+\frac{1}{n^3})^{n^2}$.

7. Find the value, when x tends to the limit 1 of the expression

   $\log(x^{5/2}-1) - \log(x^{3/2}-1)$.

8. Show that when x is small , log $\{(1+x)^{1/3}+(1-x)^{1/3}$ is approximately equal

   to $\log 2 - \frac{x^2}{9}$.

9. By using the fact that $(1+\frac{x}{n})^n = e^{x\log(1+\frac{x}{n})}$ prove that

   $(1+\frac{x}{n})^n + (1-\frac{x}{n})^{-n} = 2e^x\left\{1+\frac{1}{n^2}\left(\frac{x^2}{3}+\frac{x^4}{8}\right)\right\}$.

   Answer : 1.2, 2. $\frac{1}{10}$, 3. $\frac{3}{2}$, 4 – 1, 5. .$-\frac{1}{3}$,6. e^3, 7 $log\left(\frac{5}{3}\right)$.

Unit III

Characteristic equation – Eigen values and Eigen Vectors - Similar matrices - Cayley - Hamilton Theorem (Statement only) - Finding powers of square matrix, Inverse of a square matrix up to order 3, - related problems.

Inverse matrix:

Let A be any matrix. If a matrix B exists such that AB = BA = I, then B is called the inverse matrix of A.

Since AB and BA exist and equal to a square matrix, A and B must be square matrices of the same order.

If an inverse matrix to A exists, then it is unique. Let B and C be the inverse matrices to A.

Then AB = BA = I and AC = CA = I

Pre-multiplying AB by C we got (AB = CI)

i.e., IB = CI

i.e., B = C

The inverse of A is defined by A^{-1} is denoted by A^{-1}.

Hence $AA^{-1} = A^{-1}A = I$.

Adjoint matrix:

Let A be the square matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Then |A| is the determined of the matrix A.

Then $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

Let the cofactors of the elements $a_{11}, a_{12}, ...$ In the determinant be $A_{11}, A_{12}, ...$

Then the transpose of the matrix

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

i.e., the matrix

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Relationship between adjoint and inverse matrices:

We get

$$A(adj\ A) = \begin{bmatrix} |A| & 0 & 0 & & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

Since,

$$a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} = |A|$$

$$a_{21}A_{21} + a_{22}A_{22} + \cdots + a_{2n}A_{2n} = |A|$$

…………………………………………………

…………………………………………………

$$a_{n1}A_{n1} + a_{n2}A_{n2} + \cdots + a_{nn}A_{nn} = |A|$$

$$a_{11}A_{21} + a_{12}A_{22} + \cdots + a_{1n}A_{2n} = 0$$

…………………………………………………

$$a_{11}A_{n1} + a_{12}A_{n2} + \cdots + a_{1n}A_{nn} = 0$$

$$a_{21}A_{11} + a_{22}A_{12} + \cdots + a_{2n}A_{nn} = 0$$

$$\therefore A(adj\ A) = |A|\begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = |A|I$$

Hence $A\left(\frac{adj\ A}{|A|}\right) = I$

Similarly, we can show that $\left(\frac{adj\ A}{|A|}\right)A = I$

$\therefore \frac{adj\ A}{|A|}$ is the inverse matrix of A.

$\frac{adj\ A}{|A|}$ is also called the reciprocal of the matrix and is denoted by A^{-1}.

The inverse of A exists only when $|A| \neq 0$.

i.e., when A is non-singular.

The necessary and sufficient condition for a square matrix A to process the inverse that |A| is not zero. i.e., A is non-singular.

Let A^{-1} be the inverse of A

$\therefore AA^{-1} = I$

Hence $|A||A^{-1}| = |I| = 1$

$\therefore |A| \neq 0$ and $|A^{-1}| \neq 0$.

$\therefore$ the condition $|A| \neq 0$ is necessary.

Let $|A| \neq 0$

$$AA^{-1} = A\left\{\frac{1}{|A|}adj\ A\right\}$$

$$AA^{-1} = \frac{1}{|A|}A(adj\ A)$$

$$AA^{-1} = \frac{1}{|A|}\begin{bmatrix} |A| & 0 & 0 & & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

Similarly, $A^{-1}A = I$.

Hence the condition is sufficient.

Example 1:

$(A^T)^{-1} = (A^{-1})^T$

Let A be the matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

Then $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$

$$A^{-1} = \frac{adj\ A}{|A|} = \frac{I}{|A|}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$\therefore (A^{-1})^T = \frac{I}{|A|}\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

$$(A^T)^{-1} = \frac{adj\ A^T}{|A^T|}$$

$$(A^T)^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Since $|A^T| = |A|$

Hence $(A^T)^{-1} = (A^{-1})^T$

Inverse of AB is $B^{-1}A^{-1}$:

Let A and B be non-singular square matrices and their inverses be respectively A^{-1} and B^{-1}.

Since A and B are non-singular matrices

$|A| \neq 0; |B| \neq 0$

$\therefore |AB| \neq 0$

We have $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ (By associative law)

$AB(B^{-1}A^{-1}) = AIA^{-1}$

$AB(B^{-1}A^{-1}) = AA^{-1}$

$AB(B^{-1}A^{-1}) = I$

Similarly, $(B^{-1}A^{-1})AB = I$

$\therefore AB(B^{-1}A^{-1}) = (B^{-1}A^{-1})AB = I$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Corollary:

(i) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(ii) $(A^2)^{-1} = (A^{-1})^2$

(iii) $(A^n)^{-1} = (A^{-1})^n$

Example 2:

Find the inverse of $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$

$$|A| = 1\left(\begin{vmatrix} 8 & 2 \\ 9 & -1 \end{vmatrix}\right) - 2\left(\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix}\right) - 1\left(\begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix}\right)$$

$$|A| = 1(-8-18) - 2(-3-8) - 1(27-32)$$

$$|A| = 1(-26) - 2(-11) - 1(-5)$$

$$|A| = -26 + 22 + 5$$

$$|A| = 1 \neq 0$$

$\therefore A^{-1}$ exists

Let $adj\ A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

$$adj\ A = \begin{bmatrix} +\begin{vmatrix} 8 & 2 \\ 9 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 9 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 4 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 4 & 9 \end{vmatrix} \\ +\begin{vmatrix} 2 & 7 \\ 8 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} \end{bmatrix}$$

$$adj\ A = \begin{pmatrix} (-8-18) & -(-3-8) & (27-32) \\ -(-2+9) & (-1+4) & -(9-8) \\ (4+8) & -(2+3) & (8-6) \end{pmatrix}$$

$$adj\ A = \begin{pmatrix} -26 & 11 & -5 \\ -7 & 3 & -1 \\ 12 & -5 & 2 \end{pmatrix}^T$$

$$A^{-1} = \begin{pmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{pmatrix}$$

Example 3:

Find the inverse of the matrices $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$||A| = 1\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1(1-0) - 2(0) - 1(0) = 1 \neq 0$$

Therefore A^{-1} exists.

$$adj\ A = \begin{bmatrix} +\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\ +\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$adjA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -3 & 1 \end{bmatrix}^T$$

$$adjA = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 4:

Find the inverse of the matrices $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$|A| = 1(9-1) - (6) + 1(-2)$$

$$= 1(8) - 12 - 2 = -6 \neq 0$$

Therefore A^{-1} exists.

$$adj\ A = \begin{bmatrix} +\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$$

$$adjA = \begin{bmatrix} 8 & -6 & -2 \\ -7 & 3 & 1 \\ -5 & 3 & -1 \end{bmatrix}^T$$

$$adjA = \begin{bmatrix} 8 & -7 & -5 \\ -6 & 3 & 3 \\ -2 & 1 & -1 \end{bmatrix}$$

Hence $A^{-1} = \frac{1}{6}\begin{bmatrix} -8 & 7 & 5 \\ 6 & -3 & -3 \\ 2 & -1 & 1 \end{bmatrix}$.

Example 5:

Find the inverse of the matrices $\begin{bmatrix} 4 & 2 & 1 \\ -3 & 0 & 5 \\ -1 & 1 & 6 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 0 & 5 \\ -1 & 1 & 6 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$|A| = 4(-5) - 2(-18+5) + 1(-3)$$

$$= -20 + 26 - 3 = 3 \neq 0$$

Therefore A^{-1} exists.

$$adj\ A = \begin{bmatrix} +\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} -3 & 5 \\ -1 & 6 \end{vmatrix} & +\begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} & +\begin{vmatrix} 4 & 1 \\ -1 & 6 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ -1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ -3 & 5 \end{vmatrix} & +\begin{vmatrix} 4 & 2 \\ -3 & 0 \end{vmatrix} \end{bmatrix}$$

$$adjA = \begin{bmatrix} 5 & 13 & -3 \\ -11 & 25 & -6 \\ 10 & -23 & 6 \end{bmatrix}^T$$

$$adjA = \begin{bmatrix} 5 & -11 & 10 \\ 13 & 25 & -23 \\ -3 & -6 & 6 \end{bmatrix}$$

Hence $A^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -11 & 10 \\ 13 & 25 & -23 \\ -3 & -6 & 6 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 5/3 & -11/3 & 10/3 \\ 13/3 & 25/3 & -23/3 \\ -1 & -2 & 2 \end{bmatrix}$$

Example 6:

Find A satisfying the matrix equation

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

Solution:

Given: $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} A \begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix} = \begin{bmatrix}-2 & 4\\3 & -1\end{bmatrix}$

Let $B = \begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}$ and $C = \begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}$

$BB^{-1}(A)(CC^{-1}) = B^{-1}\begin{bmatrix}-2 & 4\\3 & -1\end{bmatrix}C^{-1}$

$A = B^{-1}C^{-1}\begin{bmatrix}-2 & 4\\3 & -1\end{bmatrix}$

$A = \begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}\begin{bmatrix}-2 & 4\\3 & -1\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$

$A = \begin{bmatrix}-4-3 & 8+1\\6+6 & -12-2\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$

$A = \begin{bmatrix}-4-3 & 8+1\\6+6 & -12-2\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$

$A = \begin{bmatrix}-7 & 9\\12 & -14\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$

$A = \begin{bmatrix}-21+45 & -14+27\\36-70 & 24-42\end{bmatrix}$

$A = \begin{bmatrix}24 & 13\\-34 & -18\end{bmatrix}$

Example 7:

Show that $A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I = 0$. Hence determine its inverse.

Solution:

$A^2 - 4A - 5I = 0$

$A^2 = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$

$$A^2 = \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiply the equation by A^{-1}

$$A^{-1}A^2 - 4AA^{-1} - 5IA^{-1} = 0$$

$$A - 4 - 5A^{-1} = 0$$

$$A - 4 = 5A^{-1}$$

$$A^{-1} = \frac{A-4}{5}$$

$$A - 4 = \begin{bmatrix} 1-4 & 2-4 & 2-4 \\ 2-4 & 1-4 & 2-4 \\ 2-4 & 2-4 & 1-4 \end{bmatrix}$$

$$A - 4 = \begin{bmatrix} -3 & -2 & -2 \\ -2 & -3 & -2 \\ -2 & -2 & -3 \end{bmatrix}$$

$$\frac{A-4}{5} = \begin{bmatrix} -\frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & -\frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} \\ -\frac{2}{5} & -\frac{3}{5} & -\frac{2}{5} \\ -\frac{2}{5} & -\frac{2}{5} & -\frac{3}{5} \end{bmatrix}$$

Example 8:

If $A = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$ find $A + A^{-1}$

Solution:

Let $A = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix}$

$|A| = 20 - 21 = -1 \neq 0.$

Therefore, A^{-1} exists.

$Adj\ A = \begin{bmatrix} 4 & -3 \\ -7 & 5 \end{bmatrix}$

$A^{-1} = -1 \begin{bmatrix} 4 & -3 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix}$

Hence $A + A^{-1} = \begin{bmatrix} 5 & 3 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 3 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 14 & -1 \end{bmatrix}$

Example 9:

If $A = \begin{bmatrix} 7 & 4 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$. Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Solution:

$$AB = \begin{bmatrix} 7 & 4 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 14+4 & 7+4 \\ -2+0 & -1+0 \end{bmatrix}$$

$$ABC = \begin{bmatrix} 18 & 11 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 54+77 & 72+99 \\ -6-7 & -8-9 \end{bmatrix} = \begin{bmatrix} 131 & 171 \\ -13 & -17 \end{bmatrix}$$

$$(ABC)^{-1} = \frac{1}{|ABC|} adj(ABC)$$

$$|ABC| = -2227 + 2223$$

$= -4$

$adj(ABC) = \begin{bmatrix} -17 & -171 \\ 13 & 131 \end{bmatrix}$

$(ABC)^{-1} = -\frac{1}{4}\begin{bmatrix} -17 & -171 \\ 13 & 131 \end{bmatrix}$

$(ABC)^{-1} = \frac{1}{4}\begin{bmatrix} 17 & 171 \\ -13 & -131 \end{bmatrix}$ …………….. (1)

$C^{-1} = \frac{1}{|C|}adj(C)$

$|C| = 27 - 28 = -1 \neq 0$

$\therefore C^{-1}$ exists

$adj(C) = \begin{bmatrix} 9 & -4 \\ -7 & 3 \end{bmatrix}$

$C^{-1} = \begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}$

$\mathrm{B}^{-1} = \frac{1}{|B|}adj(B)$

$|B| = 2 - 1 = 1 \neq 0$

$\therefore \mathrm{B}^{-1}$ exists

$adj(B) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$\mathrm{B}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$\mathrm{A}^{-1} = \frac{1}{|A|}adj(A)$

$|A| = 0 + 4 = 4 \neq 0$

$\therefore A^{-1}$ exists

$adj(C) = \begin{bmatrix} 0 & -4 \\ 1 & 7 \end{bmatrix}$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 0 & -4 \\ 1 & 7 \end{bmatrix}$$

$$C^{-1}B^{-1}A^{-1} = \frac{1}{4}\begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -4 \\ 1 & 7 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -9-4 & 9+8 \\ 7+3 & -7-6 \end{bmatrix}\begin{bmatrix} 0 & -4 \\ 1 & 7 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -13 & 17 \\ 10 & -13 \end{bmatrix}\begin{bmatrix} 0 & -4 \\ 1 & 7 \end{bmatrix}$$

$$C^{-1}B^{-1}A^{-1} = \frac{1}{4}\begin{bmatrix} 0+17 & 52+119 \\ 0-13 & -40-91 \end{bmatrix}$$

$$C^{-1}B^{-1}A^{-1} = \frac{1}{4}\begin{bmatrix} 17 & 171 \\ -13 & -131 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1)&(2) equation,

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Hence Proved.

Example 10:

Show that if $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, $A^3 - 3A^2 + 3A - 2I = 0$. Determine A^{-1}.

Solution:

$$A^2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0+0 & -1-1+0 & 0+1+0 \\ 0+0-1 & 0+1-0 & 0-1-1 \\ 1+0+1 & -1+0+0 & 0-0+1 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1-0+1 & -1-2+0 & 0+2+1 \\ -1+0-2 & 1+1-0 & 0-1-2 \\ 1-0+2 & -2-1+0 & 0+1+1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & -3 & 3 \\ -3 & 2 & -3 \\ 3 & -3 & 2 \end{bmatrix}$$

$$3A^2 = 3\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & -6 \\ 6 & -3 & 3 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & -3 \\ 3 & 0 & 3 \end{bmatrix}$$

$$2I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^3 - 3A^2 + 3A - 2I = 0$$

$$\begin{bmatrix} 2 & -3 & 3 \\ -3 & 2 & -3 \\ 3 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 3 \\ -3 & 3 & -6 \\ 6 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & -3 \\ 3 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 3 & 0 \\ 0 & -1 & 3 \\ -3 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -3 \\ 3 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$0 = 0$$

$$A^3 - 3A^2 + 3A - 2I = 0 \ \ldots\ldots\ldots\ldots\ (1)$$

$$\Rightarrow \ (1) \times A^{-1} \Rightarrow A^2 - 3A + 3I - 2A^{-1} = 0$$

$$2A^{-1} = A^2 - 3A + 3I$$

$$2A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & -3 \\ 3 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$2A^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & -2 & 1 \\ -1 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$2A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Example 11:

Show that if $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$, A satisfies the equation $A^3 - 3A^2 + 3A - 2I = 0$.

Calculate A^{-1} Solve the equation $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution:

$$A^3 = A^2.A$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-0 & 1+1+0 & 0-1+0 \\ 0+0+1 & 0+1-0 & 0-1-1 \\ -1+0-1 & -1+0+0 & 0-0+1 \end{bmatrix}$$

$$A^2.A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1+0+1 & 1+2-0 & 0-2-1 \\ 1+0+2 & 1+1-0 & 0-1-2 \\ -2+0-1 & -2-1+0 & 0+1+1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 3 & -3 \\ 3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix}$$

$$3A^2 = \begin{bmatrix} 3 & 6 & -3 \\ 3 & 3 & -6 \\ -6 & -3 & 3 \end{bmatrix}$$

$$A^3 - 3A^2 + 3A - 2I = 0$$

$$\begin{bmatrix} 2 & 3 & -3 \\ 3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -3 \\ 3 & 3 & -6 \\ -6 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 0 \\ 0 & 3 & -3 \\ -3 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & -3 & 0 \\ 0 & -1 & 3 \\ 3 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -3 \\ -3 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$0 = 0$$

Hence Proved.

$A^3 - 3A^2 + 3A - 2I = 0$ ………….. (1)

$\Rightarrow$ $(1) \times A^{-1} \Rightarrow A^2 - 3A + 3I - 2A^{-1} = 0$

$$2A^{-1} = A^2 - 3A + 3I$$

$$2A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 0 \\ 0 & 3 & -3 \\ -3 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$2A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

$$X = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 1-4-7 & 2-5-8 & 3-6-9 \\ 1+4+7 & 2+5+8 & 3+6+9 \\ 1-4+7 & 2-5+8 & 3-6+9 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} -10 & -11 & -12 \\ 12 & 15 & 18 \\ 4 & 5 & 6 \end{bmatrix}$$

Exercises 1:

1. Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ satisfies the equation $A^3 - 3A^2 + 3A - 2I = 0$. Hence determine its inverse.

2.If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, show that $A^{-1} = A^3$

3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

EIGEN VALUES AND EIGEN VECTORS:

Given a matrix A of order n, determine the scalar λ and the non-zero vectors X which simultaneously satisfies the equation

AX = λX.

Let A be $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and X be $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Hence the equation AX = λX becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

i.e., $\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_3 \end{bmatrix} - \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix} = 0.$

$(a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 = 0.$

$a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 = 0.$

$a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 = 0.$

These equations have non-trivial solutions are when

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0.$$

The expansion of the determinant gives a polynomial of degree 3 in λ which is denoted by ϕ (λ).

The equation ϕ (λ) = 0 is called the characteristic equation of the matrix A.

The root of this equation are called the characteristic value or latent values or eigen values of the matrix A.

Let P be the matrix formed by the eigen vectors x_1, x_2, x_3.

i.e., $P = [x_1, x_2, x_3]$

$AP = A[x_1, x_2, x_3]$

$= [Ax_1, Ax_2, Ax_3]$

Since x_1, x_2, x_3 satisfy the equation AX = λX when $\lambda = \lambda_1, \lambda_2, \lambda_3$ we have

$AX_1 = \lambda_1 x_1, AX_2 = \lambda_2 x_2 , AX_3 = \lambda_3 x_3.$

Therefore $AP = [\lambda_1 x_1 \ \lambda_2 x_2 , \lambda_3 x_3]$

$$= [x_1, x_2, x_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

= PD

i.e., $P^{-1}AP = D$

Here D is diagonal matrix.

This process of finding P such that $P^{-1}AP = D$ is called diagonalising the matrix A.

Note:

1. The characteristic equation of the matrix A is $[A - \lambda I] = 0$.

2. If the roots of the characteristic equation are not distinct. It may not be possible to Diagonalise the matrix A.

Corollary (i):

$AP = D$, $A = PDP^{-1}$

Corollary (ii):

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Its characteristic equation is $\begin{bmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{bmatrix} = 0$.

i.e., $(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0$.

Hence the eigen values of D are $\lambda_1, \lambda_2, \lambda_3$.

Hence A and D have the same characteristic equation and the same eigen values.

Corollary (iii):

The eigen vectors of the matrix are linearly dependent.

We have to show that if $c_1x_1 + c_2x_2 + c_3x_3 = 0$ then $c_1 = c_2 = c_3 = 0$.

Let us assume c_1, c_2, c_3 exist such that

$c_1x_1 + c_2x_2 + c_3x_3 = 0$ ………. (1)

Multiplying this equation (1) by A we get

$c_1\lambda_1A\,x_1 + c_2\lambda_2A\,x_2 + c_3\lambda_3A\,x_3 = 0$

$i.e., c_1\lambda_1 x_1 + c_2\lambda_2 x_2 + c_3\lambda_3 x_3 = 0$ ……….(2)

Multiplying this equation (2) by A we get

$c_1\lambda_1 A x_1 + c_2\lambda_2 A x_2 + c_3\lambda_3 A x_3 = 0$

$i.e., c_1\lambda_{12} x_1 + c_2\lambda_{22} x_2 + c_3\lambda_{32} x_3 = 0 \ldots\ldots\ldots\ldots(3)$

These three equations (1), (2), (3) may be written in the form

$$[c_1x_1\ c_2x_2\ c_3x_3]\begin{bmatrix}1 & \lambda_1 & \lambda_1{}^2\\ 1 & \lambda_2 & \lambda_2{}^2\\ 1 & \lambda_3 & \lambda_3{}^2\end{bmatrix} = 0. \quad \ldots\ldots\ldots\ (4)$$

If $\lambda_1, \lambda_2, \lambda_3$ are all unequal then $\begin{bmatrix}1 & \lambda_1 & \lambda_1{}^2\\ 1 & \lambda_2 & \lambda_2{}^2\\ 1 & \lambda_3 & \lambda_3{}^2\end{bmatrix} \neq 0$ and hence the matrix

B = $\begin{bmatrix}1 & \lambda_1 & \lambda_1{}^2\\ 1 & \lambda_2 & \lambda_2{}^2\\ 1 & \lambda_3 & \lambda_3{}^2\end{bmatrix}$ is non-singular and hence an inverse of the matrix exists.

If we multiply equation (4) on the right by the inverse of the matrix B, we have

$[c_1x_1\ \ c_2x_2\ \ c_3x_3] = 0.$

Since no X is zero, it implies that $c_1 = 0, c_2 = 0, c_3 = 0.$

Hence x_1, x_2, x_3 are linearly independent.

Corollary (iv):

The determinant of the matrix A is equal to the product of its eigen values and is numerically equal to the absolute term of the characteristic equation.

Let $\lambda_1, \lambda_2, \lambda_3,$ be the eigen values of the matrix then

$$\begin{vmatrix}a_{11}-\lambda & a_{12} & a_{13}\\ a_{21} & a_{22}-\lambda & a_{23}\\ a_{31} & a_{32} & a_{33}-\lambda\end{vmatrix} = -(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

Putting $\lambda = 0$ on both sides, we get $|A| = \lambda_1\lambda_2\lambda_3$

Corollary (v):

The sum of the elements on the diagonal A is the sum of the eigenvalues of the Matrix

The characteristic equation is $\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0.$

Sum of the eigenvalues = Sum of the roots of the characteristic equation

$$= \frac{Coefficient\ of\ \lambda^2}{Coefficient\ of\ \lambda^3}$$

λ^2 and λ^3 occurs only in the term $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$. When the determinant is expended.

Coefficient of $\lambda^3 = -1$

Coefficient of $\lambda^2 = a_{11}+a_{22}+a_{33}$

Hence $a_{11}+a_{22}+a_{33}$ = Sum of the eigen values of the matrix A.

Example 1:

Diagonalise the matrix $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

Solution:

The characteristic equation is,

$$\begin{vmatrix} (2-\lambda) & -2 & 3 \\ 1 & (1-\lambda) & 1 \\ 1 & 3 & (-1-\lambda) \end{vmatrix} = 0$$

$(-\lambda_3 + 2\lambda_2 + 5\lambda - 6) = 0$

$(\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$

$\therefore\ \lambda = -2,1,3.$

When $\lambda = 1$, the equation becomes,

$x_1 - 2x_2 + 3x_3 = 0$

$x_1 + x_3 = 0$

$x_1 + 3x_2 - 2x_3 = 0$

Hence $\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{5}$

Therefore, $x_1 = -1, x_2 = 1, x_3 = 1$

$\therefore X_1 = (-1,1,1)$

Similarly for the value of $\lambda = -2$, the eigen vector is

$X_2 = (11,1,-4)$

And for $\lambda = 3$, the eigenvectors $X_3 = (1,1,1)$

Hence $P = \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix}$

We can easily see that $P^{-1} = \frac{1}{30}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix}$

Hence

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix}$$

Example 2

Show that if λ is an eigenvalue of the matrix A, then λ^n is an eigenvalue of A^n, where n is a positive integer.

Solution:

Let P be the matrix with is such that $P^{-1}AP = D$ where

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A.

Hence $(P^{-1}AP)(P^{-1}AP) = D.D$

$P^{-1}A(PP^{-1})AP = D^2$

$P^{-1}AIAP = D^2$

$P^{-1}A^2P = D^2$

Multiplying this equation by $P^{-1}AP$ on both sides we get

$(P^{-1}A^2P)(P^{-1}AP) = D^2(P^{-1}AP)$

$P^{-1}A^2(PP^{-1})AP = D^2D$

$P^{-1}A^2IAP = D^3$

$P^{-1}A^3P = D^3$

Continuing this process, we get $P^{-1}A^nP = D^n$

Hence A^n and D^n have the same eigenvalues.

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

$\therefore$ The eigenvalues of D^n are $\lambda_1^n, \lambda_2^n, \lambda_3^n$.

Hence the eigenvalues of A^n are $\lambda_1^n, \lambda_2^n, \lambda_3^n$.

Similar matrices:

Two matrices A and are said to be similar of there exists a non-singular matrix P such that $P^{-1}AP = B$

If D is the diagonal matrix whose diagonal elements are the eigenvalues of the matrix A, then A and D are similar matrices.

Example 1:

If A and B are similar matrices, they have the same characteristic equation.

Since A and B are similar, a matrix P exists such that

$B = P^{-1}AP$

$\therefore B - \lambda I = P^{-1}AP - \lambda I$

$= P^{-1}AP - P^{-1}\lambda IP$

$= P^{-1}(A - \lambda I)P$

$$\text{Hence } |B - \lambda \mathrm{I}| = |\mathrm{P}^{-1}(A - \lambda \mathrm{I})\mathrm{P}|$$

$$= |P^{-1}||A - \lambda \mathrm{I}||\mathrm{P}|$$

$$= |P^{-1}||\mathrm{P}||A - \lambda \mathrm{I}|$$

$$= |P^{-1}P||A - \lambda I|$$

$$= |I||A - \lambda I|$$

$$= |A - \lambda I|$$

The characteristic equations of A and B are respectively $|A - \lambda I| = 0$ and $|B - \lambda I| = 0$.

Hence they are equal.

Corollary:

Two similar matrices have the same eigenvalues.

Cayley – Hamilton theorem:

Every matrix satisfies its characteristic equation.

Proof:

Let A be a matrix of order n.

The matrix $[A - \lambda I]$ is $\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$

Let $|A - \lambda I|$ be $\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n$.

Since $\{adj\ A\}A = |A|I$,

We have $\{adj\ [A - \lambda I]\}[A - \lambda I] = |A - \lambda I|I$

Hence $adj\ [A - \lambda I]$ is of the form

$B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}$

Where $B_0, B_1, B_2, \ldots, B_{n-1}$ are matrices of order n.

$\therefore (B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1})[A - \lambda I] = (\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n)I$

Equating the different powers of λ on both sides we get,

$B_0A = \alpha_0 I$

$B_1A - B_0 = \alpha_1 I$

$B_2A - B_1 = \alpha_2 I$

……………………

……………………

$B_{n-1}A - B_{n-2} = \alpha_{n-1}I$

$-B_{n-1} = \alpha_n I$

Multiplying these equations successively by $I, A, A^2, \ldots, A^{n-1}, A^n$ and adding we get

$\alpha_n A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I = 0$

Hence A satisfies its characteristic equation.

An important application of the Cayley-Hamilton theorem is to express the inverse of a matrix in terms of powers of A.

we have shown that

$\alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1}A^{n-1} + \alpha_n A^n = 0$

Where $\alpha_0 \neq 0$ and $|A| \neq 0$.

$\therefore \alpha_0 I = -\alpha_1 A - \alpha_2 A^2 - \cdots - \alpha_n A^n$

Per-multiplying by A^{-1}, we get

$\alpha_0 A^{-1} I = -\alpha_1 A^{-1} A - \alpha_2 A^{-1} A^2 - \cdots - \alpha_n A^{-1} A^n$

$\alpha_0 A^{-1} = -\alpha_1 I - \alpha_2 A - \cdots - \alpha_n A^{n-1}$

$\therefore A^{-1} = -\frac{\alpha_1}{\alpha_0} I - \frac{\alpha_2}{\alpha_0} A - \cdots - \frac{\alpha_n}{\alpha_0} A^{n-1}$

Higher powers of the matrices

Another important use is to calculate the higher powers of the matrices.

This is illustrated in examples 2 and 3 given below.

Example 1:

Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$ and hence determine its inverse.

Solution:

The characteristic equation is $\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$

Simplifying we get $\lambda^3 - 13\lambda + 12 = 0$.

Hence the matrix A satisfies the equation

$A^3 - 13A + 12I = 0$

Per-multiplying by A^{-1}, we have

$A^2 - 13I + 12A^{-1} = 0$

$\therefore 12A^{-1} = 13I - A^2$

$$A^2 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 31 & -18 & 11 \end{bmatrix}$$

$$12A^{-1} = 13 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 31 & -18 & 11 \end{bmatrix}$$

$$12A^{-1} = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{bmatrix} - \begin{bmatrix} 8 & 6 & 2 \\ -1 & 7 & -2 \\ 31 & -18 & 11 \end{bmatrix}$$

$$12A^{-1} = \begin{bmatrix} 5 & -6 & -2 \\ 1 & 6 & 2 \\ -31 & 18 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 5 & -6 & -2 \\ 1 & 6 & 2 \\ -31 & 18 & 2 \end{bmatrix}$$

Example 2:

If $A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$ determine A^n in terms of A.

Solution:

The characteristic equation is given by

$$\begin{vmatrix} 4-\lambda & -2 \\ 3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

Hence A satisfies the equation

$$A^2 - 7A + 6I = 0$$

Let $\lambda^n = f(\lambda)(\lambda^2 - 7\lambda + 6) + p\lambda + q$ where $\lambda = 1 \; or \; 6,\; \lambda^2 - 7\lambda + 6 = 0$

$$\therefore 1^n = p + q,\; 6^n = 6p + q$$

$$\therefore p = \frac{6^n - 1}{5},\; q = \frac{6 - 6^n}{5}$$

$$\therefore \lambda^n = f(\lambda)(\lambda^2 - 7\lambda + 6) + \frac{(6^n - 1)\lambda + (6 - 6^n)I}{5}$$

Hence $A^n = f(A)(A^2 - 7A + 6) + \frac{(6^n-1)A+(6-6^n)I}{5}$

$A^n = \frac{1}{5}[(6^n - 1)A + (6 - 6^n)I]$ since $A^2 - 7A + 6 = 0$

$$A^n = \frac{6^n - 1}{5}\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} + \frac{6 - 6^n}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 3:

Calculate A^4 when $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Solution:

The characteristic equation of the matrix A is

$$\begin{vmatrix} 1 - \lambda & 3 \\ 2 & 4 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - 5\lambda - 2 = 0$$

$$\therefore A^2 - 5A - 2I = 0$$

Hence $A^2 = 5A + 2I$

$$\therefore A^4 = (5A + 2I)(5A + 2I)$$

$$A^4 = 25a^2 + 20A + 4I$$

$$A^4 = 25(5A + 2I) + 20A + 4I$$

$$A^4 = 145A + 54I$$

$$A^4 = 145\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + 54\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 145 & 435 \\ 290 & 580 \end{bmatrix} + \begin{bmatrix} 54 & 0 \\ 0 & 54 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 199 & 435 \\ 290 & 634 \end{bmatrix}$$

Exercises 2:

1. Find the eigen vales of the following matrices:

$(i)\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

2. Find the eigen vales and the eigen vectors of the following matrices:

(i) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & 2 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

3. Diagonalise the following matrices

(i) $\begin{bmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$

Unit IV

Expansions of $\sin n\theta$, $\cos n\theta$ in powers of $\sin\theta$, $\cos\theta$ - Expansion of $\tan n\theta$ in terms of $\tan\theta$, Expansions of $\cos^n\theta$, $\sin^n\theta$, $\cos^m\theta\sin^n\theta$ –Expansions of $\tan(\theta_1+\theta_2+,\ldots,+\theta_n)$- related problems.

EXPANSIONS

Expansion of cosθ and sinθ

we have $(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n$

If n is a positive integer, the expression on the right hand side can be expanded by Binomial Theorem. Hence,

$$(\cos n\theta + i\sin n\theta) = \cos^n\theta + n\cos^{n-1}\theta\,(i\sin\theta) + \frac{n(n-1)}{2!}\cos^{n-2}\theta\,(i\sin\theta)^2 + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots\ldots$$

$i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, \ldots$

$$(\cos n\theta + i\sin n\theta) = \cos^n\theta + \frac{n(n-1)}{2!}\cos^{n-2}\theta\,(i\sin\theta)^2 + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\,(i\sin\theta)^4 + \ldots\ldots + i\left(n\cos^{n-1}\theta\sin\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\right)$$

Equating the real and imaginary parts we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \cdots$$

Note:

1. The terms are alternately positive and negative
2. Each series continues till one of the factors in the numerator is zero and then ceases.
3. The sum of the powers of cosθ and sinθ in every term of the expansions equals n.

Both the series are in descending powers of cosθ and in ascending powers of sinθ

Corollary 1:

$$\frac{\sin n\theta}{\sin\theta} = n\cos^{n-1}\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^2\theta + \cdots$$

$$= n\cos^{n-1}\theta + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta(1-\cos^2\theta) + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta$$

$$(1-\cos^2\theta^2) + \cdots$$

Similarly in the expansions of cos nθ, by putting

$$\sin^2\theta = 1-\cos^2\theta$$

Cos n θ can be expressed in a series containing powers of cos θ.

Corollary 2:

Coefficient of $\cos^{n-1}\theta$ in the expansion of

$$\frac{\sin n\theta}{\sin\theta} = n_{c_1} + n_{c_3} + n_{c_5} + \cdots = 2^{n-1}$$

Corollary 3:

Coefficient of $\cos^{n}\theta$ in the expansion of

$$\cos n\theta = n_{c_0} + n_{c_2} + n_{c_4} + \cdots = 2^{n-1}$$

Expansion of tan n θ in powers of tan θ

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$\tan n\theta = \frac{\cos^n\theta + n_{c_2}\cos^{n-2}\theta\sin^2\theta + n_{c_4}\cos^{n-4}\theta\sin^4\theta + \cdots}{n\cos^{n-1}\theta\sin\theta + n_{c_3}\cos^{n-3}\theta\sin^3\theta + \cdots}$$

On dividing both the numerator and denominator by $\cos^n\theta$

Expansion of tan (A + B + C + ...)

cosA + isin A = cos A (1 + i tan A)

cosB + i sin B = cosB (1 + i tan B)

cos C + i sin C = cos C(1 + i tan C)

∴(cos *A* + *i* sin *A*)(cos *B* + *i* sin *B*)(cos *C* + *i* sin *C*)...

= cos *A* cos *B* cos *C*...(1 + *i* tan *A*)(1 + *i* tan *B*)(1 + *i* tan *C*)...

$= \cos A \cos B \cos C...[1 + i \sum \tan A + i^2 \sum \tan A \tan B + i^3 \sum \tan A \tan B]$

$= \cos A \cos B \cos C...[1 + iS_1 - S_2 - iS_3 + ...]$

Where S_r is the sum of products taken r at a time of tan A, tan B, tan C,…

Equating the real and imaginary parts on both sides, we have

$\cos(A + B + C + ...) = \cos A \cos B \cos C...(1 - S_2 + S_4 + ...)$

$\sin(A + B + C + ...) = \cos A \cos B \cos C...(S_1 - S_3 + S_5 + ...)$

$\therefore \tan(A + B + C + ...) = \frac{S1 - S3 + S5 + ...}{1 - S2 + S4 + ...}$

Corollary:

Putting $A = B = C = ... = \theta$ taking n angles

Where S_r is the sum of the products taken r at a time of tan A, tan A, …, tan A n terms

Hence S1=tan θS2= n_{c_2} tan ^{2} θ , $S_3 = n_{c_3}$ $\tan^3 \theta$ …..

$$\tan n\theta = \frac{n_{c_1} \tan\theta - n_{c_2} \tan^3\theta + \cdots}{1 - n_{c_2} \tan^2\theta + n_{c_4} \tan^4\theta + \cdots}$$

Example 1:

Express $\cos 8\theta$ in terms of sinθ

Solution:

$\cos 8\theta + i \sin 8\theta = (\cos 8\theta + i \sin 8\theta)^8$

$= cos^8\theta + 8_{c_1} cos^7\theta(isin\theta) + 8_{c_2} cos^6\theta(isin\theta)^2 + \cdots$

$= cos^8\theta - 8_{c_2} cos^6\theta sin^2\theta + 8_{c_4} cos^4\theta sin^4\theta - 8_{c_6} cos^2\theta sin^6\theta + 8_{c_8}\ i(8_{c_1} cos^7\theta sin\theta +$

$8_{c_3} cos^5\theta sin^3\theta + 8_{c_5} cos^3\theta sin^5\theta - 8_{c_7} cos\theta sin^7\theta$

Equating the real parts, we have

$\cos 8\theta = cos^8\theta - 8_{c_2} cos^6\theta sin^2\theta + 8_{c_4} cos^4\theta sin^4\theta - 8_{c_6} cos^2\theta sin^6\theta + 8_{c_8}\ sin^8\theta.$

$\cos 8\theta = (1 - sin^2\theta)^4 - 28(1 - sin^2\theta)^3 sin^2\theta + 70(1 - sin^2\theta)^2 sin^4\theta - 28(1 -$
$sin^2\theta) sin^6\theta +. sin^2\theta$

$\cos 8\theta = (1 - 4sin^2\theta + 6sin^4\theta - 4sin^6\theta + sin^8\theta) - 28(1 -$
$3sin^2\theta + 3sin^4\theta - sin^6\theta) sin^2\theta) + 70(1 - 2sin^2\theta + sin^4\theta) sin^4\theta - 28(1 -$
$sin^2\theta) sin^6\theta + sin^8\theta$

$\cos 8\theta = (1 + 28 + 70 + 28 + 1) sin^8\theta + (-4 - 84 - 140 - 28) sin^6\theta + (6 + 84 +$
$70) sin^4\theta + (-4 - 28) sin^2\theta + 1$

Example 2:

Express $\frac{sin6\theta}{sin\theta}$ in terms of cos θ

Solution:

$\cos\ 6\theta + i \sin 6\theta = (\cos\ \theta + i\sin\theta)^6$

$=cos^6\theta + 6_{c_1}cos^5\theta(isin\theta) + 6_{c_2}cos^4\theta(isin\theta)^2 + 6_{c_3}cos^3\theta(isin\theta)^3 + 6_{c_4}cos^2\theta(isin\theta)^4 + 6_{c_5}cos\theta(isin\theta)^5 + (isin\theta)^6$

$=cos^6\theta + 6_{c_2}cos^4\theta sin^2\theta + 6_{c_4}cos^2\theta sin^4\theta - sin^6\theta + i(6_{c_1}cos^5\theta sin\theta - 6_{c_3}cos^3\theta sin^3\theta + 6_{c_5}cos\theta sin^5\theta$

Equating the imaginary parts on both sides,

$sin6\theta = 6_{c_1}cos^5\theta sin\theta - 6_{c_3}cos^3\theta sin^3\theta + 6_{c_5}cos\theta\ sin^5\theta$

$sin6\theta = 6cos^5\theta sin\theta - 20cos^3\theta sin^3\theta + 6\ cos\theta\ sin^5\theta$

$\frac{sin6\theta}{sin\theta} = 6cos^5\theta - 20cos^3\theta sin^2\theta + 6\ cos\theta\ sin^4\theta$

$=6cos^5\theta - 20cos^3\theta(1 - cos^2\theta) + 6(1 - cos^2\theta)^2$

$=32cos^5\theta - 32cos^3\theta + \ 6\ cos\theta$

Example 3:

If $\alpha, \beta\ and\ \gamma\ be\ the\ roots\ of\ the\ equcation\ x^3 + px^2 + qx + P = 0$. Prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radius except when q=1.

Solution:

$x^3 + px^2 + qx + P = 0.$

$\alpha + \beta + \ \gamma = -p,$

$\alpha\beta + \beta\gamma + \ \alpha\gamma = q,$

$\alpha\beta\gamma = -p$

$\tan^{-1}\alpha = x_1, \tan^{-1}\beta = x_2, \tan^{-1}\gamma = x_3$

$$\alpha = tanx_1, \beta = tanx_2, \gamma = tanx_3$$

$$tanx_1 + tanx_2 + tanx_3 \Rightarrow -p \qquad , s_{1=}p$$

$$tanx_1tanx_2 + tanx_2tanx_3 + tanx_1tanx_3 \Rightarrow +q \qquad , s_{2=}q$$

$$tanx_1tanx_2tanx_3 \Rightarrow -p \qquad , s_{3=} - p$$

$$\tan(x_1 + x_2 + x_3) = \frac{s_3 - s_1}{1 - s_2} = \frac{-p + p}{1 - q}$$

$$\tan(x_1 + x_2 + x_3) = \ 0$$

$$(x_1 + x_2 + x_3) = n\pi$$

$$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$$

Example 4:

Prove that the equation $\frac{ab}{cos\theta} - \frac{bc}{sin\theta} = a^2 - b^2$ has four roots and that the sum of the 4 values of θwhich satisfy it is equal to an odd multiple of radius π

Solution:

We know that $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}$, $\cos\theta = \frac{1-tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}$

$\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

Where t= tan θ

$$\frac{ah(1+t^2)}{1-t^2} - \frac{bk(1+t^2)}{2t} = a^2 - b^2$$

2tah+(1+t^2)-bk(1+t^2)(1-t^2)=$a^2 - b^2(2t)(1-t^2)$

2tah+$2t^3ah - bk(1-t^4) = 2t(a^2-b^2) - 2t^3(a^2-b^2)$

2tah+$2t^3ah - bk + bkt^4 - 2ta^2 + 2tb^2 + 2t^3a^2 - 2t^3b^2 = 0$

Bk$t^4 + 2t^3(ah + a^2 - b^2) + 2t(ah - a^2+b^2) - bk = 0$

Let $t_1, t_2, t_3, and\ t_4,$ be the

$s_1 = \frac{-2}{1} = -2 \quad s_3 = \frac{-2}{1} = -2$

$$s_2 = \frac{0}{1} = 0 \quad s_4 = \frac{-11}{1} = -11$$

$\tan(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}) = \frac{s_1 - s_3}{1-s_2+s_4}$

denominator, 1-0-1=0

$\tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}\right) = \infty$

$$\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} = (2n+1)\frac{\pi}{2}$$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n+1)\pi$$

Example 5:

Find the equation whose roots are $2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$, $2cos\frac{6\pi}{7}$

Solution:

By previous sum,

$\text{Sin}7\theta = 7sin\theta - 56sin^3\theta + 112sin^5\theta - 64sin^7\theta$

$\frac{sin7\theta}{sin\theta} = 7 - 56sin^2\theta + 112sin^4\theta - 64sin^6\theta$

$=7 - 28(1 - cos2\theta) + 28(1 - cos2\theta)^2 - 8(1 - cos2\theta)^3$

We know that

$\cos2\theta = 1 - 2sin^2\theta$

$\sin2\theta = \frac{1 - cos2\theta}{2}$

$\frac{sin7\theta}{sin\theta} = 7 - 28 + 28cos2\theta + 28 + 28cos^2 2\theta - 56cos2\theta - 8 + 8cos^3 2\theta - 24cos^2 2\theta + 24cos2\theta$

$\frac{sin7\theta}{sin\theta} = 8cos^3 2\theta + 4cos^2 2\theta - 4cos2\theta - 1$

$put\ x = cos2\theta$

$\frac{sin7\theta}{sin\theta} = 8x^3 + 4x^2 - 4x - 1$

Where $\theta = \pm\frac{\pi}{7}, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7}$

$\sin7\theta = 0$

$8cos^3 2\theta + 4cos^2 2\theta - 4cos2\theta - 1 = 0$

Has the roots, $\pm\frac{\pi}{7}, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7}$

Put $\cos2\theta = x\ we\ get,$

$8x^3 + 4x^2 - 4x - 1 = 0\ has\ the\ roots\ \cos\pm\frac{2\pi}{7}, \cos\pm\frac{4\pi}{7}, cos\pm\frac{6\pi}{7}$

Let y=2x

$y^3 - y^2 - 2y - 1 = 0\ has\ the\ roots\ 2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2cos\frac{6\pi}{7}$

Example 6:

Show that $\cos\frac{\pi}{9}, \cos\frac{2\pi}{9}, cos\frac{4\pi}{9} = \frac{1}{8}$.

Solution:

We know that

$\cos 9\theta = 256cos^9\theta - 576cos^7\theta + 432cos^5\theta - 120cos^3\theta + 9\cos\theta$

where $\theta = 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{12\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

therefore, $1=256cos^9\theta - 576cos^7\theta + 432cos^5\theta - 120cos^3\theta + 9\cos\theta$

put x= $\cos\theta$

$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$ has the roots,

$\cos 0 = 1, cos\ \frac{2\pi}{9}, cos\ \frac{4\pi}{9}, cos \frac{6\pi}{9}, cos \frac{8\pi}{9}, cos\ \frac{10\pi}{9}, cos \frac{12\pi}{9}, cos \frac{14\pi}{9}, cos\ \frac{16\pi}{9}$

since 1 is a root

(x -1) is the factor

$256x^8 + 256x^7 + 320x^6 - 320x^5 + 112x^4 + 112x^3 - 8x^2 - 8x+1=0$

has the roots , $cos\ \frac{2\pi}{9}, cos\ \frac{4\pi}{9}, cos \frac{6\pi}{9}, cos \frac{8\pi}{9}, cos\ \frac{10\pi}{9}, cos \frac{12\pi}{9}, cos \frac{14\pi}{9}, cos\ \frac{16\pi}{9}$,

$cos\ \frac{18\pi}{9}, cos\ \frac{20\pi}{9}$

The equation $\cos(2\pi - \theta) = cos\theta$

$256x^8 + 256x^7 + 320x^6 - 320x^5 + 112x^4 + 112x^3 - 8x^2 - 8x+1=0$

Has the roots

$cos^2\left(\frac{2\pi}{9}\right), cos^2\left(\frac{4\pi}{9}\right), cos^2\left(\frac{6\pi}{9}\right), cos^2\left(\frac{8\pi}{9}\right)$

Taking square root on both

$16x^4 + 8x^3 - 12x^2 - 4x+1=0$

$\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, cos\frac{6\pi}{9}, cos\frac{8\pi}{9}$

$cos\ \frac{6\pi}{9} = \cos(\frac{2\pi}{3}) = \cos(3-1)\frac{\pi}{3}$

$= cos\ \frac{3\pi}{3} - cos\ \frac{\pi}{3}$

$=\cos(\pi - \frac{\pi}{3}) = -cos\frac{\pi}{3}$

$=\frac{-1}{2}$

The equation has the roots $\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, cos\frac{8\pi}{9}, \frac{-1}{2}$

X=$\frac{-1}{2}$, $2x + 1$ is a factor of the eqn

$\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, cos\frac{8\pi}{9}$

$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}cos\frac{8\pi}{9} = \frac{-1}{2}$

$$\cos\frac{8\pi}{9} = \cos(\pi - \frac{\pi}{9}) = -cos\frac{\pi}{9}$$

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}cos\frac{4\pi}{9} = \frac{1}{8}$$

Example 7:

Find the equation whose roots are $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, tan\frac{3\pi}{5}$and $\tan\frac{4\pi}{5}$

Solution:

We know that

$\tan 5\theta$ $=\tan\theta - 5c_3 tan^3\theta + 5c_5 tan^5\theta$

$1 - 5c_2 tan^2\theta + 5c_4 tan^4\theta$

where $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

$\tan 5\theta = 0$

$5\tan\theta - 5c_3 tan^3\theta + 5c_5 tan^5\theta = 0$

Has the roots when $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

Put $\tan\theta = x$

$5x - 10x^3 + x^5 = 0$ has the roots $tan0, tan\ \frac{\pi}{5}, tan\frac{2\pi}{5}, tan\frac{3\pi}{5}, tan\frac{4\pi}{5}$

Since 0 is the roots of the equation we have

$x^4 - 10x^2 + 5 = 0$ has the roots, $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, tan\frac{3\pi}{5}$and $\tan\frac{4\pi}{5}$

Example 8:

Prove that $\tan\frac{\pi}{11}\tan\frac{2\pi}{11}tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11}=\sqrt{11}$

Solution:

We know that

$$tan11\theta = \frac{nc_1\,tan\theta - nc_3 tan^3\theta + nc_5 tan^5\theta - nc_7 tan^7\theta + nc_9 tan^9\theta - nc_{11} tan^{11}\theta}{1 - 11c_2 tan^2\theta + 11c_4 tan^4\theta - 11c_6 tan^6\theta + 11c_8 tan^8\theta - 11c_{10} tan^{10}\theta}$$

where $\theta = 0, \frac{\pi}{11}, \frac{2\pi}{11}, \frac{3\pi}{11}, \frac{4\pi}{11}, \frac{5\pi}{11}, \frac{6\pi}{11}, \frac{7\pi}{11}, \frac{8\pi}{11}, \frac{9\pi}{11}, \frac{10\pi}{11}$

If we put $tan11\theta = 0, we\ get\ the\ eqn$

$$11tan\theta - 11c_3 tan^3\theta + \cdots tan^{11}\theta = 0 \qquad \rightarrow (1)$$

has roots $tan\theta, where\ \theta\ is\ 0, \frac{\pi}{11}, \frac{2\pi}{11}, \frac{3\pi}{11}, \frac{4\pi}{11}, \frac{5\pi}{11}, \frac{6\pi}{11}, \frac{7\pi}{11}, \frac{8\pi}{11}, \frac{9\pi}{11}, \frac{10\pi}{11}$

Put tan$\theta = x$ then the eqn (1) reduces to

$$11x - 165x^3 + 462x^5 - 330x^7 + 55x^9 - x^{11} = 0 \quad \rightarrow (2)$$

Hence equation (2) has roots 0,

$tan\frac{\pi}{11}, tan\ \frac{2\pi}{11}, tan\frac{3\pi}{11}, tan\frac{4\pi}{11}, tan\ \frac{5\pi}{11}, tan\frac{6\pi}{11}, tan\frac{7\pi}{11}, tan\ \frac{8\pi}{11}, tan\frac{9\pi}{11}, tan\ \frac{10\pi}{11}$

since

$$tan\ \frac{10\pi}{11} = -\ tan\frac{\pi}{11}, \quad tan\frac{9\pi}{11} = -tan\frac{2\pi}{11}, \quad tan\frac{8\pi}{11} = -tan\frac{3\pi}{11}, \quad tan\frac{7\pi}{11}$$

$$= -tan\frac{4\pi}{11}, \quad tan\frac{6\pi}{11} = -tan\frac{5\pi}{11}$$

$$x^{10} - 55x^8 + 330x^6 - 462x^4 + 165x^2 - 11x = 0 \quad \rightarrow (3)$$

Has roots $\pm tan\frac{\pi}{11}, \pm tan\ \frac{2\pi}{11}, \pm tan\frac{3\pi}{11}, \pm tan\frac{4\pi}{11}, \pm tan\ \frac{5\pi}{11}$

put $x^2 = y$ then the eqn (3) reduce to

$$y^5 - 55y^4 + 330y^3 - 462y^2 + 165y - 11 = 0 \quad \rightarrow (4)$$

This equation has roots,

$tan^2\frac{\pi}{11}, tan^2\ \frac{2\pi}{11}, tan^2\frac{3\pi}{11}, tan^2\frac{4\pi}{11}\ : tan^2\frac{\pi}{11}tan^2\ \frac{2\pi}{11}tan^2\frac{3\pi}{11}tan^2\frac{4\pi}{11} = 11$

$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11} = \sqrt{11}$

The negative sign is discarded

Since all the terms of the expression on the left side are positive, each angle involved being a side.

Example 9:

Expand tan4θ in terms of tan θ and show that tan$\frac{\pi}{16}$, tan$\frac{5\pi}{16}$, $tan\frac{9\pi}{16}$, tan$\frac{13\pi}{16}$ are roots of the equation.

Solution:

$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

tan$4\theta = \frac{s_1 - s_3}{1 - s_2 + s_4} = \frac{4c_1\ tan\theta - 4c_3 tan^3\theta}{1 - 4c_2 tan^2\theta + 4c_4 tan^4\theta}$

$1 = \frac{4x - 4c_3x^3}{1 - 4c_2x^2 + x^4}$

$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16},$

$\tan(4\theta) = \tan(\frac{\pi}{4})\tan(\frac{5\pi}{4})\tan(\frac{9\pi}{4})\tan(\frac{13\pi}{4})$

$x^4 - 6x^2 + 1 - 4x + 4x^3 = 0$

Powers of sines and cosines of θ in terms of function of multiples of θ.

Let $\cos\theta + i\sin\theta = x$

then $\cos\theta - i\sin\theta = \frac{1}{x}$

Adding , $2\cos\theta = x + \frac{1}{x}$ ……………(1)

Subtracting, , $2i\sin\theta = x - \frac{1}{x}$ ……………(2)

$x^n=(cos\theta + isin\theta)^n = cosn\theta + isinn\theta$

$$\frac{1}{x^n} = (cos\theta + isin\theta)^{-n} = cosn\theta - isinn\theta$$

$x^n+\frac{1}{x^n} = 2cosn\theta$ ……………(3)

$x^n-\frac{1}{x^n} = 2isinn\theta$ ……….......(4)

We make use of three relation (1)(2)(3) and (4) to expand $cos^n\theta$ and$sin^n\theta$ in series of cosines and sines of multiples of θ.

Expansion of $cos^n\theta$ when is a positive integer

$2\cos\theta = x + \frac{1}{x}$

$(2cos\theta)^n=(x + \frac{1}{x})^n$

$$=x^n + nc_1x^{n-1}\frac{1}{x} + nc_2x^{n-2}\frac{1}{x^2} + \cdots \ldots\ldots nc_{n-2}x^2\frac{1}{x^{n-2}} + nc_{n-1}x\frac{1}{x^{n-1}}\frac{1}{x^n}$$

$$= x^n+\frac{1}{x^n} + nc_1(x^{n-2}+\frac{1}{x^{n-2}}) + nc_2(x^{n-4}+\frac{1}{x^{n-4}}) + \cdots$$

Since

$x^n+\frac{1}{x^n} = 2\cos n\theta\,, we\ have$

$2^n cos^n\theta = 2\cos n\,\theta + nc_1\, 2\cos(n-2)\,\theta + nc_2\, 2\cos(n-4)\theta + ...$

$2^{n-1} cos^n\theta = \cos n\,\theta + nc_1\, 2\cos(n-2)\,\theta + nc_2\, 2\cos(n-4)\theta + ...$

Note:

1) If n is odd there will be (n+1) terms in the expansion of $\left(x+\frac{1}{x}\right)^n$ and hence these can be grouped in pairs. Hence the last term contains cosθ.We can easily see that the coefficient of cosθ in the expression of
   $2^{n-1}cos^n\theta\ is\ independent\ of\ \theta\ and\ is\ equal\ to\ \frac{1}{2}nc_{\frac{(n-1)}{2}}$
2) When n is even, the number of terms in the expansion of $\left(x+\frac{1}{x}\right)^n$ is (n +1) and the middle term is independent of x and is left over when all the other terms are grouped in pairs hence the last term in the expansion of $2^{n-1}cos^n\theta\ is\ independent\ of\ \theta\ and\ is\ equal\ to\ \frac{1}{2}nc_{\frac{n}{2}}$

Example 1:

Expand $\cos^6\theta$ and $\cos^5\theta$ in series of cosines of multiples of θ

Solution:

Let x=$\cos\theta + i\sin\theta$

Then $(2cos\theta)^6 = (x+\frac{1}{x})^6$

$$=x^6 + 6c_1x^5\frac{1}{x} + 6c_2x^4\frac{1}{x^2} + 6c_3x^3\frac{1}{x^3} + 6c_4x^2\frac{1}{x^4} + 6c_5x\frac{1}{x^5}$$

$$=(x^6+\frac{1}{x^6}) + 6c_1(x^4+\frac{1}{x^4}) + 6c_2(x^2+\frac{1}{x^2}) + 6c_3 + \cdots$$

$$=2cos6\theta + 6c_1(2cos4\theta) + 6c_2(2cos2\theta) + 6c_3$$

$2^5\cos^6\theta = cos6\theta + 6(cos4\theta) + 15(cos2\theta) + 10$

$\cos^6\theta=\frac{1}{32}(cos6\theta + 6(cos4\theta) + 15(cos2\theta) + 10)$

again $(2cos\theta)^5 = (x+\frac{1}{x})^5$

$$=x^5 + 5c_1x^4\frac{1}{x} + 5c_2x^3\frac{1}{x^2} + 5c_3x^2\frac{1}{x^3} + 5c_4x\frac{1}{x^4} + \frac{1}{x^5}$$

$$=(x^5+\frac{1}{x^5}) + 5c_1(x^3+\frac{1}{x^3}) + 5c_2(x+\frac{1}{x})$$

$$=2cos5\theta + 5c_1(2cos3\theta) + 5c_2(2cos\theta)$$

$2^4\cos^5\theta = cos5\theta + 5(cos3\theta) + 10(cos\theta)$

Expansion of $sin^n\theta$ when is a positive integer

$$2\text{isin}\,\theta = x - \frac{1}{x}$$

$(2isin\theta)^n = (x-\frac{1}{x})^n$

$$=x^n - nc_1x^{n-1}\frac{1}{x} + nc_2x^{n-2}\frac{1}{x^2} - nc_3x^{n-3}\frac{1}{x^3} + \cdots$$

Case (1) n is even

The number of turn in the expansion is odd. The sings of the term are alternatively positive and negative and the last term is positive.

$(2isin\theta)^n = (x^n+\frac{1}{x^n}) - nc_1(x^{n-2}+\frac{1}{x^2}) + nc_2(x^{n-4}+\frac{1}{x^4})$

(i.e.)$(2^n)(-1)^{\frac{n}{2}}sin^n\theta = (2cosn\theta) - nc_1 2\cos(n-2)\theta + nc_2\cos(n-4)\theta$ .....

Hence,

$\left((-1)^{\frac{n}{2}}(2^{n-1})\right)sin^n\theta = (cosn\theta) - nc_1\cos(n-2)\theta + nc_2\cos(n-4)\theta$ ...

case (2) n is odd

$(2isin\theta)^n = x^n - nc_1x^{n-2} + nc_2x^{n-4}$ ..... $-\frac{1}{x^n}$

$$=(x^n+\frac{1}{x^n}) - nc_1(x^{n-2}+\frac{1}{x^{n-2}}) + nc_2(x^{n-4}+\frac{1}{x^{n-4}})$$

$=(2isinn\theta) - nc_1 2isin(n-2)\theta + nc_2\, 2\mathrm{isin}(n-4)\,\theta$

(i.e.) $2^{n-1}(i)^{n-1}sin^n\theta = (sinn\theta) - nc_1 sin(n-2)\theta + nc_2 \sin(n-4)\,\theta$ ... ..

(i.e.) $2^{n-1}(i)^{\frac{n-1}{2}}sin^n\theta = (sinn\theta) - nc_1 sin(n-2)\theta + nc_2 \sin(n-4)\,\theta$ ... ..

Example 1:

Expand $sin^7\theta$ in a series of sines of multiples of θ

Solution:

we have,

$$(x-\frac{1}{x})^7 = x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{1}{x^5} - \frac{1}{x^7}$$

$$=(x^7 - \frac{1}{x^7}) - 7(x^5 - \frac{1}{x^5}) + 21(x^3 + \frac{1}{x^3}) - 35(x - \frac{1}{x})$$

putting x=$\cos\theta + i\sin\theta$

so that , $x^n - \frac{1}{x^n} = 2\mathrm{isin}\, n\theta$ for all integral values of n, we have

$(2\mathrm{isin}\,\theta)^7 = 2\mathrm{isin}\,7\theta - 7(2\mathrm{isin}\,5\theta) + 21(2\mathrm{isin}\,3\theta) - 35(2\mathrm{isin}\,\theta)$

(i.e.) $2^6(-1)^3 sin^7\theta = \sin 7\theta - 7(\sin 5\theta) + 21(\sin 3\theta) - 35(\sin\theta)$

$$sin^7\theta = \frac{-1}{64}(sin7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$$

Example 2:

Expand $sin^6\theta$ in a series of sines of multiples of θ

Solution:

we have,

$$(x-\frac{1}{x})^6 = x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

$$=(x^6 - \frac{1}{x^6}) - 6(x^4 - \frac{1}{x^4}) + 15(x^2 + \frac{1}{x^2}) - 20$$

putting x=cos $\theta + i \sin\theta$ $\quad x - \frac{1}{x} = 2isin\theta \;\; and \; x^n + \frac{1}{x^n} = 2cosn\theta$ for all integer value of n

$(2\text{isin}\,\theta)^6 = 2\cos 6\,\theta - 6(\,2\cos 4\,\theta) + 15(2\cos 2\,\theta) - 20$

(i.e.) $2^6\,(-1)^3\, sin^6\theta\; = 2\cos 6\theta - 6(\,2\cos 4\,\theta) + 15(\cos 2\theta) - 20$

$$sin^6\theta = \frac{-1}{32}(cos6\theta - 6\cos 4\,\theta + 15\cos 2\theta - 10$$

Example 3:

Expand $sin^3\theta cos^5\theta\; in$ a series of of sines of multiples of θ

Solution:

$(2isin\theta)^3(2cos\theta)^5 = (x - \frac{1}{x})^3(x + \frac{1}{x})^5$

$= (x^2 - \frac{1}{x^2})^3(x + \frac{1}{x})^5$

$= (x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6})(\,x^2 + 2 + \frac{1}{x^2})$

$= (x^8 - \frac{1}{x^8}) + 2(x^6 - \frac{1}{x^6}) - 2(x^4 + \frac{1}{x^4}) - 6(x^2 - \frac{1}{x^2})$

$= 2\text{isin}\,8\theta + 2(\,2\text{isin}\,6\theta) - 2(2\text{isin}\,4\theta) - 6(2\text{isin}\,2\theta)$

(i.e.) $2^3\,(-i)\, sin^3\theta 2^5 cos^5\theta\; = 2i(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta)$

$$sin^3\theta cos^5\theta = \frac{-1}{2}(sin8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta)$$

Example 4:

Expand $sin^4\theta cos^2\theta\; in$ a series of sines of multiples of θ

Solution:

$(2isin\theta)^4(2cos\theta)^2 = (x - \frac{1}{x})^4(x + \frac{1}{x})^2$

$= (x^2 - \frac{1}{x^2})^2(x + \frac{1}{x})^2$

$=(x^4 - 2 + \frac{1}{x^4})(x^2 - 2 + \frac{1}{x^2})$

$=(x^6\text{-} \frac{1}{x^6}) - 2(x^4\text{-}\frac{1}{x^4}) - 2(x^2\text{+}\frac{1}{x^2}) + 4$

$=2\cos 6\,\theta - 2(\,2\cos 4\,\theta) - 2(2\cos 2\,\theta) + 4$

(i.e.) $2^4\ sin^4\theta 2^2 cos^2\theta \quad =2\cos 6\theta - 2(\,2\cos 4\,\theta) - 2(\cos 2\theta) + 4$

$sin^4\theta cos^2\theta = \dfrac{1}{26}(cos6\theta - 2\cos 4\,\theta - 2\cos 2\theta + 4)$

UNIT V

Hyperbolic functions – Relation between circular and hyperbolic functions Inverse hyperbolic functions, Logarithm of complex quantities, - related problems.

Hyperbolic Function

Introduction:

If θ is expressed in Radians, $\cos\theta$ and $\sin\theta$ can be expanded is powers of θ, the Result Begins

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\infty \rightarrow (1)$$
$$\sin\theta = 1 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\infty \rightarrow (2)$$

(These expansions are valid for all values of. θ, real or Imaginary.)

The student is familiar with the exponential series, for all values of x.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\cdots$$

where

$$e = 1 + \frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{1}{n!} +$$

Put $x = i\theta$ in (3) then,

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots\infty$$
$$= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots\infty$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\infty\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\cdots\infty\right)$$
$$= \quad \cos\theta + i\sin\theta \text{ from (1) and (2)}$$

(this, formula is known as Euler's Formulas)

Put $x = -i\theta$ in (3) Then

$$e^{-i\theta} = 1 + \frac{(-i\theta)}{1!} + \frac{(-i)^2}{2!} + \frac{(-i\theta)^3}{3!} + \cdots \infty$$
$$= 1 - \frac{i\theta}{1!} - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} \dots \infty$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right) - i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots \infty\right)$$
$$= \cos\theta - i\sin\theta$$

Hence we get the Relation

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding $2\cos\theta = e^{i\theta} + e^{-i\theta}$

ie, $\cos\theta = \frac{e^{-i\theta} + e^{+i\theta}}{2}$ ……….(4)

Subtracting we get the relation

$$2i\sin\theta = e^{-\theta} - e^{-i\theta}$$
$$ie)\ \sin\theta = \frac{e^{-i\theta} - e^{-i\theta}}{2!} \qquad \dots \dots .. (5)$$

Hyperbolic Function:

The expression $\frac{1}{2}(e^x + e^{-x})$ and $\frac{1}{2}(e^x - e^{-x})$ are defined as hyperbolic cosines and hyperbolic sine Respectively of the angle x and symbolically.

$$\cos hx = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}$$

The hyperbolic tangent, secant cosecant and cotangent are obtained from the hyperbolic sine and cosine. Just as the ordinary tangent, secant, cosecant and cotangent are obtained from the ordinary sine and cosine.

Thus,

$$\tan hx = \frac{\sinh x}{\cos hx}, \sec hx = \frac{1}{\cosh x}$$

$$\operatorname{cosec} hx = \frac{1}{\sin hx}, \coth x = \frac{1}{\tan hx}$$

Relation Between Hyperbolic Functions:

(1) $\cosh^2 x - \sinh^2 x = \frac{1}{4}\{(e^x + e^x)^2 - (e^x - e^{-x})^2\} = 1$

(2) $2\sinh x\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right) \cdot \left(\frac{e^x + e^{-x}}{2}\right)$

$$= \frac{(e^{2x} - e^{-2x})}{2}$$
$$= \sinh 2x.$$

(3) $\cos h^2 x + \sinh^2 x \quad = \{1/4(e^x + e^{-x})^2 + (e^x - e^{-x})^2\}$

$$= 1/4(e^{2x} + 2 + e^{-2x}) + (e^{2x} - 2 + e^{-2x})\}$$
$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x.$$

(4) From the relation. (3), we get the Relations

$$\cosh 2x = 2\cosh^2 x - 1$$
$$\cos h2x = 1 + 2\ln h^2 x$$
$$\cos h^2 x = 1/2(\cos h2x + 1)$$
$$\sinh h^2 x = 1/2(\cosh 2x - 1)$$

(5) The series for sinh x and cosh x derived Below.

$$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$$
$$e^{-x} = 1 - x + x^2/2! - x^3/3! + \cdots$$

Subtracting $e^x - e^{-x} = 2(x + x^3/3! \cdots \infty)$

$\therefore \sin hx = x + x^3/3! + x^5/5! \cdots \infty$

Adding, $e^x + e^{-x} = 2\left(1 + \frac{x^2}{2}! + \frac{x}{4}! + \cdots\right)$

$\therefore \cosh x = 1 + x^2/2! + x^4/4! + \cdots \infty$

(6) we have seen that

$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{2}}{2}$$

Put $\theta = ix$ in these relations We have

$$\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cos hx.$$
$$\sin(ix) = \frac{e^{-x} - e^{x}}{2} = (i)^2 \frac{\sin hx}{i}$$
$$= i\sin hx.$$
$$\therefore \tan(ix) = i\tan hx.$$

The following Relation also hold good :-

$$\sinh(i0) = \frac{e^{i0} - e^{-10}}{2} = 1\sin\theta$$
$$\cos h(i\theta) = \frac{e^{i0} + e^{-10}}{2} = \cos\theta$$

$\tan h(i\theta) = i\tan\theta$

Using these relation, we can derive relation between hyper functions corresponding to rotation Between circular functions.

For example,

(i) $\sin^2\theta + \cos^2\theta = 1$, put $\theta = ix$

$\therefore \sin^2(ix) + \cos^2(ix) = 1$

i.e., $(i\sin hx)^2 + (\cos hx)^2 = 1$ i.e., $\cosh^2 x - \sinh^2 x = 1$

(ii)

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos(2ix) = \cos^2(ix) - \sin^2(ix)$$
$$= (\cosh x) - (i\sin hx)^2$$
$$\therefore \cosh 2x = \cosh^2 x + \sinh^2 x$$

(iii) $\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$\sin(2ix) = 2\sin(ix)\cos(ix)$

(i.e.)$\sin h2x = 2i\sinh h\cosh x$

(i.e.) $\sin h2x = 2\sin hx\cosh x$

(iv)

$$1 + \tan\theta = \sec^2\theta$$
$$1 + \tan^2(ix) + \mathrm{cec}^2(1x)$$
$$1 + (i\tan h)^2 = \frac{1}{(\cosh)^2}$$
$$(\mathrm{i.e})\ 1 - \tan h^2 x + \mathrm{sech}^2 x$$

(v) $\sin(\theta + \varphi) = \sin\theta\cos p + \cos\theta\sin^2\theta$

Put $\theta = ix,\ p = iy$ then.

$\sin(ix + iy) = \sin(ix)\cos(iy) + \cos(ix)\sin(iy)$

(i.e.) $\sinh(x + y) = i\sinh x\cosh y +$ (cosh (isis by)

$\therefore \sinh(x + y) = \sinh x\cosh y + \cosh x\ \sin hy$

Similarly

$\sin h(x - y) = \sin hx\cos hy - \cosh x\sin x$

$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

Put $\theta = ix, \phi = iy$, then

$$\cos(ix + iy) = \cos ix\cos iy \cdot \sin ix\sin iy$$
$$\cos h(x + y) = \cos hx\cos hy - (i\sin hx)(i\sinh y)$$

$= \cos hx\cosh y + \sinh x\sin hy.$

Similarly

$\cos h(x-y) = \cos hx \cos hy - \sin hx \sinh y$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Rut $\theta = ix$

$$\therefore \tan(2ix) = \frac{2\tan(ix)}{1-\tan^2(ix')}$$
$$i\tan h2x = \frac{2i\tan hx}{1-(i\tan hx)^2}$$

$$\tan h2x = \frac{2\tan hx}{1+\tanh h^2 x}$$

Inverse hyperbolic Functions:

We can express $\sin h^{-1}x$, $\cos h^{-1}x, \tan h^{-1}x$ inters of logarithmic functions

(i) Let $y = \sin h^{-1}x$; Then $x = \sin hy$

$\therefore 1/2(e^y - e^{-y}) = x$

(i.e.) $e^{2y} - 1 = 2xe^y$

(i.e.) $e^{2y} - 2xe^y - 1 = 0$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since e^Y is always positive

$e^y = x + \sqrt{x^2+1}$

Taking logarithms to the Base e on both sides, we get",

$$y = \log_e\left(x+\sqrt{x^2+1}\right)$$
$$\because \sinh h^{-1}x = \log_e\left(x+\sqrt{x^2+1}\right)$$

(ii) $y = \cosh^{-1} x$ then $x = \cos hy$.

$\therefore \frac{1}{2}(e^y + e^{-y}) = x$

$e^{2y} - 2xe^y + 1 = 0$

$$\therefore e^y = x \pm \sqrt{x^2 - 1}$$
$$= x + \sqrt{x^2 - 1} \text{ (or)}$$
$$= \frac{1}{x\sqrt{x^2 - 1}}$$

$\therefore y = \log_e\left(x + \sqrt{x^2 - 1}\right)$ or

$-\log_e\left(x + \sqrt{x^2 - 1}\right)$

$= \pm\log_e\left(x + \sqrt{x^2 - 1}\right)$

The positive sign is usually taken

$\cos h^{-1}x = \log_e\left(x + \sqrt{x^2 - 1}\right)$

Let $y = \tan^{-1} x$ Then

$x = \tan hy$

$\therefore \frac{e^y - e^{-y}}{e^y + e^{-y}} = x$

$\dot{u}, e^y - e^{-y} = (x)(e^y + e^{-y})$

$\text{ie}, e^y + e^y(1 - x)$

$e^{2y} = \frac{1 + x}{1 - x}$

$2y = \log_e\left(\frac{1 + x}{1 - x}\right)$

(i.e.) $y = 1/2\log e(1 + x/1 - x)$

$\therefore \tanh^{-1} x = 1/2\log e(1 + x/1 - x)$.

Example 1:

If $\cosh u = sec\ \theta$, show that $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

Solution:

Let $\cosh u = sec\ \theta$

$\mathrm{u} = \cos \mathrm{h}^{-1}(sec\ \theta)$

$= \log_e(\sec\theta + \sqrt{\sec^2 - 1}$

$= \log_e(\sec\theta + \tan\theta)$

$$= \log_e\left(\frac{1 + \sin\theta}{\cos\theta}\right)$$

$$= \log_e\left\{1 + \frac{2\tan\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2}}\right\} \div \left\{\frac{1 - tan^2\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2}}\right\}$$

$$= \log_e\frac{1 - tan^2\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2}}$$

$$= \log_e\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

Example 2:

If $\tan A = \tan \propto \tan h\beta \tan B \cos\alpha \tan h\beta$, Prove that

$\tan\ (A + B) = \mathrm{sinh2}\ \beta\mathrm{cosec}\ 2\mathrm{\alpha}$.

Solution:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$= \frac{\tan\alpha\tan\beta + \cot\alpha\tan h\beta}{1 - \tan\alpha\tan h\beta \cdot \cot\alpha\tan h\beta}$$

$$= \frac{\tan h\beta(\tan\alpha + \cot\alpha)}{1 - \tanh^2\beta}$$

$$= \frac{\sin h\beta\cos h\beta}{\cos h^2\beta - \sinh h^2\beta}\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right)$$

$$= \frac{\sin h\beta\cos h\beta}{\sin\alpha\cos\alpha\sin\beta}$$

$$= \frac{1/2\sin 2\beta}{1/2\sin h2\alpha}$$

$$= \sin h2\beta\operatorname{cosec} 2\alpha.$$

Example 3:

Express $\cos h^6\ \theta$ in terms of hyperbolic cosines of multiples of θ.

Solution:

$$\cos h^6\theta = \left(\frac{e^{\theta} + e^{-\theta}}{2}\right)$$

$$= \frac{1}{2^6}\left[e^{6\theta} + 6c_1e^{4\theta} + 6c_2e^{2\theta} + 6c_3 + 6c_4e^{-2\theta} + 6c_5e^{-4\theta} + 6c_6e^{-6\theta}\right]$$

(by binomial theorem)

$$= \frac{1}{2^6}[(e^{6\theta} + e^{-6\theta}) + 6c_1(e^{4\theta} + e^{-4\theta}) + 6c_2(e^{2\theta} + e^{-2\theta}) + 6c_3]$$

$$= \frac{1}{2^6}[\cosh 6\theta + 6.\cosh 4\ \theta + 15\cosh 2\theta + 10]$$

Example 4:

If $\cos\alpha\cosh\beta = \cos\phi, \sin\alpha\sinh\beta = \sin\phi,$

Prove that $\sin\phi = \pm\sin^2\alpha = \pm\sinh^2\beta$

Solution:

$$\cosh\beta = \frac{\cos\phi}{\cos\alpha}, \sinh\beta = \frac{\sin\phi}{\sin\alpha}$$

we know that $\cosh^2\beta - \sinh^2\beta = 1$

$$\left(\frac{\cos\phi}{\cos\alpha}\right)^2 - \left(\frac{\sin\phi}{\sin\alpha}\right)^2 = 1$$
$$\cos^2\phi\sin^2\alpha = \sin^2\phi\cos^2\alpha = \sin^2\alpha\cos^2\alpha$$
$$(1-\sin^2\phi)\sin^2\alpha - \sin^2\phi(1-\sin^2\alpha) = \sin^2\alpha(1-\sin^2\alpha)$$

$$\sin^2\alpha - \sin^2\phi\sin^2\alpha - \sin^2\phi + \sin^2\phi\sin^2$$
$$= \sin^2\alpha - \sin^4\alpha$$

$$-\sin^2\phi = -\sin^4\alpha$$
$$\sin^2\phi = \sin^4\alpha$$
$$\sin\phi = \pm\sin^2\alpha.$$

We have, $\sin\phi = \pm\sin^2\alpha$

$\sin\alpha\sinh\beta = \pm\sin^2\alpha$

$\sinh\beta = \pm\sin\alpha$

Taking square on both sides

$\sinh^2\beta = \pm\sin^2\alpha = \sin\phi$

Example 5:

If $\cos(x+iy) = \cos\theta + i\sin\theta$, prove that$\cos 2x + \cos h2y = 2$

Solution:

$$\cos\theta + i\sin\theta = \cos(x+iy)$$
$$= \cos x\cos iy - \sin x\sin iy$$
$$= \cos x\cosh y - i\sin x\sinh y$$

Equating real & Imaginary parts

$$\cos\theta = \cos x \cosh y$$
$$\sin\theta = -\sin x \sinh y$$
$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos^2 x \cosh^2 y + \sin^2 x \sin h^2 y = 1$$

$$\cos^2 x \cosh y + (1-\cos^2 x)(\sin^2 hy) = 1$$
$$\cos^2 x \cosh y + \sin^2 y - \cos^2 x \sinh^2 y = 1$$
$$\cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh h^2 y = 1$$
$$\frac{\cos 2x + 1}{2} + \frac{\cosh 2y - 1}{2} = 1$$
$$\cos 2x + \cosh 2y = 2.$$

Example 6:

If $\sin(A+iB) = x+iy$ prove that

i) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

$ii)\ \dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$

Solution:

$$x + iy = \sin(A+iB)$$
$$= \sin A \cos iB + \cos A \sin 1B$$
$$= \sin A \cosh B + \cos A i \sinh B$$

Equating real & Imaginary part

$$x = \sin A \cosh B$$
$$y = \cos A \sinh B$$

$$i)\quad \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \frac{\sin^2 A \cos^2 hB}{\sin^2 A} - \frac{\cos^2 A \sin^2 hB}{\sin^2 A}$$
$$= \cos^2 hB - \sin^2 hB = 1$$

$$\text{ii)}\ \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1 = \frac{\sin^2 A \cos^2 hB}{\cos^2 hB} + \frac{\cos^2 A \sin^2 hB}{\sin^2 hB}$$
$$= \sin^2 A + \cos^2 A = 1.$$

Example 7:

If $\tan(x+iy) = u+iv$, prove that $u/v = \frac{\sin 2x}{\sinh 2y}$.

Solution:

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$
$$= \frac{2\cos(x-iy)\sin(x+iy)}{2\cos(x-iy)\cos(x+iy)}$$
$$= \frac{\sin(2x)+\sin(2iy)}{\cos 2x+\cos 2iy}$$
$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x+\cosh 2y}$$

This expression is given as u+iv

$$\therefore u = \frac{\sin 2x}{\cos 2x+\cosh 2y}$$

$$u = \frac{\sinh 2y}{\cos 2x+\cosh 2y}$$
$$u/v = \frac{\sin 2x}{\sinh 2y}.$$

Example 8:

If $\cos h(a+ib)\cos(c+id) = 1$, prove that

1) $\cos b\cos c\cos ha\cos hd + \sin b\ \sin c\sin$ ha $\sin hd = 1$

2.) $\tan ha\tan b =$ tan, hd $\tan c$

Solution:

$$1 = \cosh(a+ib)\cos(c+id)$$
$$= \{\cosh a\cosh(ib) + \sin hasinhb\}$$

$\{\cos c\cos(id) - \sin c\sin(id)$

$\cos h(iy) = \cos y$

and $\sin h(iy) = i\sin y$

$\therefore \cos h(ib) = \cos b$and $\sin h(ib)$

$= 1\sin b$

substitute these values in equation (1), we have.

$1 = (\cos ha\cos b + i\sinh a\sin b)(\cos c\ \cos hd - i\sin c\sinh d\ )$

$$= (\cos h\, a \cos b \cos c\ + \sinh a \sin b \sinh d)$$

$$+\mathrm{i}(\sin ha \sin b\ \cos c\ \cos hd - \cos ha \cos b\, \mathrm{sin}c\, \sinh d\ )$$

Equating the real parts, we get the result (1)

Equating the imaginary parts, we have

$\sin ha \sin b\ \cos c\ \cos hd - \cos ha \cos b\, \mathrm{sin}c \sinh d = 0$

$$\frac{\sin h\ a \sin b}{cos\ ha\ cosb} - \frac{\sin c \sin hd}{cos\ c\ coshd} = 0$$

$\tan ha \tan b - \tan c \tan hd = 0$

Example 9:

Separate into real and imaginary parts $\tan h\,(1 + \mathrm{i})$

Solution:

$tan\ (ix) = i\ tanhx$

$$\therefore i\tan h(1+1) = \frac{\sin(i-1)}{\cos(i-1)}$$

$$= \frac{2\cos(i+1)\sin(i-1)}{2\cos(i+1)\cos(i-1)}$$
$$= \frac{\sin(2i) - \sin(2)}{\cos(2i) + \cos(2)}$$
$$= \frac{i\sinh 2 - \sin 2}{\cosh 2 + \cos(2)}$$

$$\therefore \tan h(i+1) = \frac{\sin h(2) + i\sin 2}{\cos h(2) + \cos(2)}$$

Example 10:

Separate into real and imaginary parts $\tan^{-1}(x+iy))$

Solution:

Let $\tan^{-1}(x+iy) = \alpha + i\beta$

Then $\tan(\alpha + i\beta) = x + iy$

we easily see' that $\tan(\alpha + i\beta) \ = x + iy$

$$\tan 2\alpha = \tan(\alpha + i\beta + \alpha - i\beta)$$
$$= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta)\tan(\alpha - i\beta)}$$

$$= \frac{\alpha + iy + x - iy}{1 - (x+iy)(x-iy)}$$
$$\tan 2\alpha = \frac{2x}{1 - (x^2 + y^2)}$$
$$\alpha = 1/2\tan^{-1}\left(\frac{2x}{1 - x^2 - y^2}\right)$$
$$\tan(2\beta i) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$$

is $i\tanh 2\beta = \frac{\tan(\alpha+i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha+i\beta)\tan(\alpha-i\beta)}$

$$= \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+x^2+y^2}$$

$$\tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$$\beta = \frac{1}{2}\tan h^{-1}\left(\frac{2y}{1+x^2+y^2}\right).$$

Exercises:

1. Prove that $\cosh 2x = \frac{1+tanh^2x}{1-tanh^2x}$
2. Prove that $\sin h\, 3x = 3\, sinhx + 4 \sin h^3$
3. Prove that $\tan\ h\ 3x = \frac{tanh^3x+3\ tanhx}{1+3\tan h^2x}$
4. Prove that $\cot h^{-1}x = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$
5. Prove that $\tan h^{-1}\left(\frac{x^2+1}{x^2-1}\right) = logx\ (x>0)$

Logarithms of Complex quantities:

Definition:

If u and z be any two complex quantities such that $z = e^u$, then u is called the logarithm of z and we write $u = \log_e z$ or simply $\log z$

To find the logarithm of $x + iy$

Let $\log_e(x+iy) = \alpha + i\beta$

Then by definition

$$x + iy = e^{\alpha+i\beta}$$
$$= e^\alpha \cdot e^\alpha(\cos\beta + i\sin\beta)$$
$$x = e^\alpha\cos\beta, y = e^\alpha\sin\beta$$

Hence $e^{2\alpha} = x^2 + y^2\ \alpha = \frac{1}{2}\log(x^2+y^2)$

and $\tan\beta = \frac{y}{x} + \beta\tan^{-1}\frac{y}{x}$

$\log_e\left(x + (y) = \frac{1}{2}(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\right.$

$= \log \gamma + i\theta$

Where $r = \sqrt{x^2 + y}$ and $\tan\theta = \frac{y}{x}$

This the real part of the logarithm of a complex quantity is the logarithm of its modulus and the imaginary part is its amplitude.

General value of logarithm of x + iy

let $\log_e(x + iy) = \alpha + i\beta$

Then $x + iy = e^{\alpha + i\beta}$

$$\begin{aligned}
&= e^{\alpha} \cdot e^{i\beta} \\
&= e^{\alpha(\cos\beta + i\sin\beta)} \\
&= e^{\alpha\{\cos(2n\pi+\beta) + i\sin(2n\pi+\beta)} \\
&= e^{\alpha} \cdot e^{i\{(2n\pi+\beta)} \\
&= e^{\alpha + 2n\pi i + i\beta}
\end{aligned}$$

It follows from the definition that $\alpha + i\beta + 2n\pi i$ is the value of $\log_e(x + iy)$ This is called the general value and is written with a capital letter

$$\begin{aligned}
&\log_e(x + iy) = \alpha + i\beta \\
&\log_e(x + iy) = \log_e(x + iy) + 2b\pi i
\end{aligned}$$

It is thus clear that the logarithm of a complex quantity has more than one value. It is easy to note that the several values of $\log(x + iy)$ differ. from one another by an integral multiple of $2\pi i$

$$\log_e(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}(x) + 2n\pi i$$

Corollary 1: Put $y = 0$

Then $\log x = \frac{1}{2}\log(x^2) + 2n\pi i = \log x + 2n\pi i$

Hence the logarithm of a real positive quantity is many valued and that the principal value of the logarithm in its ordinary logarithm which is real.

Corollary 2:

Let $y = 0$ and x be negative (say, $-x,$)

if $\log(x + \mathrm{i}y) - \alpha + i\beta$, then

$x - e^{\alpha}\cos\beta, y = e^{\alpha}\sin\beta$

In this case, $e\alpha\cos\beta = -x$, and

$$e^{\alpha}\sin\beta = 0$$
$$x^2 = e^{2\alpha} \;\therefore e^{\alpha} = x$$

$\cos\beta = -1$ and $\sin\beta = 0$

$$\beta = \pi$$
$$\log(-x_1) = \log x_1 + i\pi$$
$$\log(-x_1) = \log x_1 + i(2n+1)\pi$$

Hence the principal value of the logarithm of a negative quantity is imaginary corollary 3.

Put $x = 0$

$$\log_e(iy) = \frac{1}{2}\log(y^2) + i\tan^{-1}(\infty) + 2n\pi i$$
$$= \log y + i\frac{\pi}{2} + 2n\pi i$$
$$= \log y + i\left(2n + \frac{1}{2}\right)\pi$$

Hence the logarithm of a purely imagining quantity consists of two Parts one real part and other imaginary.

Example 1:

Find $\log(1 - i)$

Solution:

$$\log(1-i) = \log(1-i) + 2n\pi i$$
$$= \frac{1}{2}\log\{1^2 + (-1)^2\} + i\tan^{-1}\left(\frac{-1}{i}\right) \cdot 2n\pi i$$
$$= \frac{1}{2}\log 2 + i\tan^{-1}(-1) + 2n\pi i$$
$$= \frac{1}{2}\log 2 + i\frac{3\pi}{4} + 2n\pi i$$
$$= \frac{1}{2}\log 2 + i\left(2n\pi + \frac{3\pi}{4}\right)$$

Example 2:

If $\log\sin(\theta + i\phi) = L + iB$, prove that $2e^{2L} = \cos h2\phi - \cos 2\theta$

Solution:

$$L + iB = \log\sin(\theta + i\phi)$$
$$= \log(\sin\theta\cos i\phi + \cos\theta\sin i\phi)$$
$$= \log(\sin\theta\cosh\phi + i\cos\theta\sinh\phi)$$
$$= \frac{1}{2}\log\left\{(\sin\theta\cosh\phi)^2 + \frac{(\cos\theta\sinh\phi}{\sin\theta\cosh\phi}\right)$$
$$L = \frac{1}{2}\log[(\sin\theta\cosh\phi)^2 + (\cos\theta\sinh\phi)^2\}$$
$$e^{2L} = (\sin\theta\cosh\phi)^2 + (\cos\theta\sinh\phi)^2$$
$$= \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi$$
$$0 = \frac{1-\cos 2\theta}{2} \cdot \cosh h^2\phi + \frac{1+\cos 2\theta}{2} \cdot \sinh^2\theta$$
$$1 = \frac{1}{2}\{(\cosh^2\phi + \sinh h^2\phi) - (\cosh^2\phi - \sinh^2\phi)\}$$
$$2e^{2L} = \cos^2\phi^2 + \sinh^2\phi - \cos 2\theta(\cosh^2\phi - \sinh^2\phi\}$$
$$= \cosh 2\phi - \cos 2\theta$$

Example 3:

Deduce the expansion of $\tan^{-1} x$ in powers of x from the expansion of $\log(a + i^{\circ}b)$

Solution:

$$\log(a + ib) = \frac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$$

Put $a = 1, b = x$

$$\log(1 + ix) = \frac{1}{2}\log(1 + x^2) + \tan^{-1}(x)$$

$\tan x =$ imaginery part of $\log(1 + ix)$

$=$ imagine ry Part $(ix) - \frac{1}{2}(ix)^2 + \frac{1}{3}(ix)^3 - \frac{1}{4}(ix)^4 + \frac{1}{5}(ix)^5 + \cdots$

$=$ imagery Part $(ix) + \frac{1}{2}x^2 - \frac{1}{3}ix^3 - \frac{1}{4}x^4 + \frac{1}{5}i^5 \ldots$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \ldots$$

Example 4:

Reduce $(\alpha + i\beta)^{x+iy}$ to the term $A + iB$

Solution:

$$\begin{aligned}(\alpha + i\beta)^{(x+iy)} &= e^{(x+iy)\log(\alpha+i\beta)} \\ &= e^{(x+iy)\{\log(\alpha+i\beta)+2n\pi i\}} \\ &= e^{(x+iy)}\{\log\gamma + i'\theta + 2\pi\pi i\}\end{aligned}$$

Where $\gamma = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

$$\begin{aligned}(\alpha + i\beta)^{x+iy} &= e^{x\log\gamma - y(\theta+2n\pi)} \cdot e^{i}\{y\log\gamma + x(\theta + 2n\pi)\} \\ &= e^{x\log\gamma - y(\theta+2n\pi)}[\cos\{y\log\gamma + x(\theta + 2n\pi)\} + i\sin\{y\cos g\,\gamma + x(\theta + 2n(\pi)\}]\end{aligned}$$

$$A = e^{x\log\gamma - y(\theta+2n\pi)}[\cos\{y\log\gamma + x(\theta + 2n\pi)\}] \; and \; B = e^{x\log\gamma - y(\theta+2n\pi)\}}$$

Example 5:

Show the $\log_i \dot{L} = \frac{4n+1}{4m+1}$, where m and n are integers.

Solution:

Let $\log_i i = x + iy$

Then $i = i^{x+iy}$

Taking the general value of the logarithm on both sides, we have

$$(x + iy)\log i = \log i$$
$$x + iy = \frac{\log_e i}{\log e^i}$$
$$= \frac{\left(2n + \frac{1}{2}\right)\pi i}{\left(2m + \frac{1}{2}\right)\pi i^{\circ}}$$
$$= \frac{4n + 1}{4m + 1}$$

When n and m are integers.

Example 6:

Find the general value of $\log_{(-3)}(-2)$

Solution:

Let $\log_{(-3)}(-2) = x + y$

$$(x + iy) = \frac{\log_e(-2)}{\log_e(-3)}$$
$$(x + iy)\log_e(-3) = \log_e(-2)$$
$$(x + iy)\log' 3 + i(2m + 1)\pi y = \log 2 + i(2n + 1)\pi$$

equating the real and imaginary Parts on both sides we get.

$x\log 3 - y(2m + 1)\pi = \log_2$

$y\log 3 + x(2m + 1)\pi = (2n + 1)\pi$

Solving the equations (1) and (2) we get

$$x = \frac{(2m+1)(2n+1)\pi^2 + (\log 2)(\log 3)}{(\log 3)^2 + (2m+1)^2\pi^2}$$

and $$y = \frac{\log 3(2n+1)\pi - (2m+1)\pi \log 2}{(\log 3)^2 + (2m+1)^2\pi^2}$$

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